Algebra: Hungerford Page 96

David Joseph Stith

1. Theorem. *If $N \triangleleft G$ and N, G/N are both p-groups, then G is a p-group.*

Let $g \in G$ so that $gN \in G/N$. Then since G/N is a p-group, the order of gN is a power of some prime p.

Therefore

$$\begin{aligned} (gN)^{p^k} = N \quad \text{for some } k \in \mathbf{Z} &\Longrightarrow g^{p^k}N = N \\ &\Longrightarrow g^{p^k} \in N \\ &\Longrightarrow o(g^{p^k}) = p^l \quad \text{for some } l \in \mathbf{Z} \text{ since } N \text{ is a } p\text{-group} \\ &\Longrightarrow (g^{p^k})^{p^l} = e \\ &\Longrightarrow g^{p^k p^l} = g^{p^{k+l}} = e \\ &\Longrightarrow o(g) | p^{k+l} \quad \text{by Theorem 3.4} \end{aligned}$$

Therefore since g is arbitrary in G, and since the order of g is a power of p, G is a p-group.

Therefore if $N \triangleleft G$ and N, G/N are both p-groups, then G is a p-group. **Q.E.D.**

10. Theorem. *Every group of order 12, 28, 56, and 200 must contain a normal Sylow subgroup, and hence is not simple.*

Let G be a group such that $|G| = 12 = 2^2 \cdot 3$. Let s_p be the number of Sylow p-subgroups for each prime divisor p. Then

$$\begin{aligned} s_2 \equiv 1 \pmod 2 \quad &\text{and} \quad s_2 | 12 \\ s_3 \equiv 1 \pmod 3 \quad &\text{and} \quad s_3 | 12 \end{aligned}$$

Therefore $s_2 \in \{1, 3\}$ and $s_3 \in \{1, 4\}$. We will show that either s_2 or s_3 is 1. Then it will follow that the single Sylow p-subgroup is normal in G.

Suppose $s_3 = 4$. Then for any two distinct Sylow 3-subgroups, H and K, $H \cap K = \{e\}$ since

$$\begin{aligned} |H \cap K| \text{ divides } |H| = 3 &\Longrightarrow |H \cap K| \in \{1, 3\} \\ &\Longrightarrow |H \cap K| = 1 \text{ since } H \neq K \\ &\Longrightarrow |H \cap K| = \{e\} \text{ since } e \text{ is contained in every subgroup} \end{aligned}$$

Therefore each of the 4 Sylow 3-subgroups contains 2 elements of order 3 not contained in any other Sylow 3-subgroup. Therefore there are 8 elements of order 3, leaving only 4 elements that could be contained in the Sylow 2-subgroups (since the order each element of a subgroup must divide the order of the subgroup). Therefore since each Sylow 2-subgroup has order 4, there can only be one of them.

Therefore either $s_3 = 1$ or $s_3 = 4 \Longrightarrow s_2 = 1$. Therefore there exists a normal subgroup in G. Therefore every group of order 12 must contain a normal Sylow subgroup and hence is not simple.

Now let G be a group such that $|G| = 28 = 2^2 \cdot 7$. Then,

$$s_7 \equiv 1 \pmod 7 \quad \text{and} \quad s_7 | 28$$

Therefore since 1, 2, and 4 are the only divisors of 28 that are not multiples of 7, and among these only $1 \equiv 1 \pmod 6$, consequently $s_7 = 1$ so that this single Sylow 7-subgroup is normal in G. Therefore every group of order 28 must contain a normal Sylow subgroup and hence is not simple.

Now let G be a group such that $|G| = 56 = 2^3 \cdot 7$. Then,

$$\begin{aligned} s_2 &\equiv 1 \pmod 2 \quad \text{and} \quad s_2 | 56 \\ s_7 &\equiv 1 \pmod 7 \quad \text{and} \quad s_7 | 56 \end{aligned}$$

Therefore $s_2 \in \{1, 7\}$ and $s_7 \in \{1, 8\}$.

We will show that either s_2 or s_7 is 1. Then it will follow that the single Sylow p-subgroup is normal in G. Suppose $s_7 = 8$. Then for any two distinct Sylow 7-subgroups, H and K, $H \cap K = \{e\}$ since

$$\begin{aligned} |H \cap K| \text{ divides } |H| = 7 &\Longrightarrow |H \cap K| \in \{1, 7\} \\ &\Longrightarrow |H \cap K| = 1 \text{ since } H \neq K \\ &\Longrightarrow |H \cap K| = \{e\} \text{ since } e \text{ is contained in every subgroup} \end{aligned}$$

Therefore each of the 8 Sylow 7-subgroups contains 6 elements of order 7 not contained in any other Sylow 7-subgroup. Therefore there are 48 elements of order 7, leaving only 8 elements that could be contained in the Sylow 2-subgroups (since the order each element of a subgroup must divide the order of the subgroup). Therefore since each Sylow 2-subgroup has order 8, there can only be one of them.

Therefore either $s_7 = 1$ or $s_7 = 8 \Longrightarrow s_2 = 1$. Therefore there exists a normal subgroup in G. Therefore every group of order 56 must contain a normal Sylow subgroup and hence is not simple.

Now let G be a group such that $|G| = 200 = 2^3 \cdot 5^2$. Then,

$$s_5 \equiv 1 \pmod 5 \quad \text{and} \quad s_5 | 200$$

Therefore since 1, 2, 4, and 8 are the only divisors of 200 that are not multiples of 5, and among these only $1 \equiv 1 \pmod 5$, consequently $s_5 = 1$ so that this single Sylow 5-subgroup is normal in G. Therefore every group of order 200 must contain a normal Sylow subgroup and hence is not simple. **Q.E.D.**

11. Problem. *How many elements of order* 7 *are there in a simple group of order* 168*?*

Let G be a simple group such that $|G| = 168 = 2^3 \cdot 3 \cdot 7$. Since G is simple, there must be more than 1 Sylow 7-subgroup. Then,

$$s_7 \equiv 1 \pmod 7 \quad \text{and} \quad s_7 | 168$$

implies that $s_7 = 8$ since 1, 2, 3, 4, 6, 8, 12, and 24 are the only divisors of 168 that are not multiples of 7, among these only 1 and 8 are congruent to $1 \pmod 7$, and $s_7 \neq 1$.

Then for any two distinct Sylow 7-subgroups, H and K, $H \cap K = \{e\}$ since

$$\begin{aligned} |H \cap K| \text{ divides } |H| = 7 &\Longrightarrow |H \cap K| \in \{1, 7\} \\ &\Longrightarrow |H \cap K| = 1 \text{ since } H \neq K \\ &\Longrightarrow |H \cap K| = \{e\} \text{ since } e \text{ is contained in every subgroup} \end{aligned}$$

Therefore each of the 8 Sylow 7-subgroups contains 6 elements of order 7 not contained in any other Sylow 7-subgroup. Therefore there are at least 48 elements of order 7 in G. But there can be no elements of order 7 in G other than those contained in the Sylow 7-subgroups since if $a \in G$ such that $o(a) = 7$ then $a \in \langle a \rangle$ and $\langle a \rangle$ is a Sylow 7-subgroup.

Therefore there are exactly 48 elements of order 7 in a simple group of order 168.

13. Theorem. *Every group G of order p^2 (p prime) is abelian.*

Let G be a group of order p^2 where p is prime. We will show that the center of G:

$$C(G) = \{g \in G : xg = gx \quad \forall x \in G\}$$

is all of G by showing that $|C(G)| = p^2$. Then it will follow that G is abelian.

By Corollary II.5.4 we know that $|C(G)| > 1$. Therefore by Lagrange $|C(g)| \in \{p, p^2\}$. We need to show that $|C(G)| = p^2$. Suppose to the contrary that $|C(G)| = p$ and, since $C(G) \lhd G$, consider the quotient group $G/C(G)$. We have

$$|G/C(G)| = \frac{|G|}{|C(G)|} = \frac{p^2}{p} = p$$

so that $G/C(G)$ is cyclic. Now let $N = C(G)$. Then $G/C(G) = \langle gN \rangle$ for some $g \in G$ and hence any element of $G/C(G)$ can be expressed as $(gN)^x = g^x N$ for some $x \in \mathbf{Z}$. We will show that $ab = ba$ for all $a, b \in G$ and hence $|C(G)| = |G| = p^2$.

Suppose $a, b \in G$. Then $a = g^x n_1$ and $b = g^y n_2$ for some $x, y \in \mathbf{Z}$ and $n_1, n_2 \in N$. Then,

$$\begin{aligned} ab &= g^x n_1 g^y n_2 \\ &= n_1 g^x g^y n_2 \quad \text{since } n_1 \in N = C(G) \\ &= n_1 g^{x+y} n_2 \\ &= g^{x+y} n_1 n_2 \\ &= g^{y+x} n_2 n_1 \\ &= g^y g^x n_2 n_1 \\ &= g^y n_2 g^x n_1 \quad \text{since } n_2 \in N = C(G) \\ &= ba \end{aligned}$$

Therefore $|C(G)| = |G| = p^2$. Therefore G is abelian.

Therefore every group G of order p^2 (p prime) is abelian. **Q.E.D.**