Modern Physics: Chapter 1 Homework

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Section 1. *Special Relativity*

1. *(book #1) If the speed of light were smaller than it is, would relativistic phenomenon be more or less conspicuous than they are now?*

The fact that relativistic phenomenon would occur at lesser speeds, if the speed of light were smaller than it is, suggests that relativistic phenomenon would be **more** conspicuous than now. However, this may be subjective, since our observational methods are ultimately dependent upon the speed of light as well and would be altered likewise.

Section 2. *Time Dilation*

2. *An athlete runs the 10-meter dash at* 10 *m/s. How much time will her watch gain or lose, as compared to ground-based clocks, during the race?*

We assume that the 10 meter distance and the 10 m/s speed were measured in the earth's reference frame so that the dash appeared to ground-based observers to take exactly 1 second. However, since we are measuring an event occurring in a moving frame, the time for us appears dilated, so that we have $t = 1s$, $v = 10$ m/s, and the time interval in seconds measured by the runner herself is t_0 given by

$$\begin{aligned} t_0 &= t\sqrt{1 - \frac{v^2}{c^2}} \\ &\approx t\left(1 - \frac{v^2}{2c^2}\right) \text{ using the binomial approximation.} \\ &= 1 - \frac{50}{c^2} \text{ seconds} \\ &\approx 1 - 5.56 \times 10^{-16} \text{ seconds} \end{aligned}$$

Therefore the 1 second interval measured on the ground corresponds to 5.56×10^{-16} seconds less than 1 second on the runner's own watch.

3. *An observer on a spacecraft moving at* $0.600c$ *relative to the earth finds that a car takes* 50.0 *minutes to take a trip. How long does the trip take to the driver of the car?*

Since the trip appears to the observer to take 50.0 minutes but is taking place in a different reference frame, the proper time is unknown and the apparent time $t = 50.0$ minutes is a dilation of the proper time. Then we have $v = 0.600c$ and the time t_0 in minutes from the driver of the car's perspective is given by

$$\begin{aligned} t_0 &= t\sqrt{1 - v^2/c^2} \\ &= (50.0)\sqrt{1 - 0.360} \text{ minutes} \\ &= (50.0)(0.800) \text{ minutes} \\ &= 40.0 \text{ minutes} \end{aligned}$$

Therefore the trip takes 40.0 minutes according to the driver of the car.

4. *How fast must a spacecraft travel relative to the earth for each day on the spacecraft to correspond to 4 days on the earth?*

   We have $t = 4$ days, $t_0 = 1$ day, and we must find v. Hence we have,

$$\begin{aligned} t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}} &\Longrightarrow 1 - \frac{v^2}{c^2} = \frac{t_0^2}{t^2} \\ &\Longrightarrow v^2 = c^2\left(1 - \frac{t_0^2}{t^2}\right) \\ &\Longrightarrow v = c\sqrt{1 - \frac{t_0^2}{t^2}} \\ &\Longrightarrow v = \frac{\sqrt{15}}{4}c \approx 0.968c \end{aligned}$$

Therefore the spacecraft must travel at a speed of approximately $0.968c$ relative to the earth for each day on the spacecraft to correspond to 4 days on the earth.

Section 4. *Length Contraction*

5. *(book #17) An astronaut whose height on earth is exactly 6 feet is lying parallel to the axis of a spacecraft moving at $0.90c$ relative to the earth. What is his height as measured by an observer in the same spacecraft? By an observer on the earth?*

   The astronaut's height measured by an observer in the same spacecraft is still 6 feet since the measurement is taken at rest. The apparent height in feet measured by an observer on earth, however, is L given by:

$$\begin{aligned} L &= L_0\sqrt{1 - \frac{v^2}{c^2}} \text{ where } L_0 = 6 \text{ feet, } v = 0.90c. \\ &= 6\sqrt{1 - 0.81} \text{ feet} \\ &\approx 2 \text{ feet } 7 \text{ inches} \end{aligned}$$

6. *A meter stick is moving with speed* $0.8c$ *relative to a frame* S. *(a) What is the stick's length, as measured by observers in* S, *if the stick is parallel to its velocity* v? *(b) What if the stick is perpendicular to* v? *(c) What if the stick is at* 60 *degrees to* v, *as seen in the stick's rest frame?*

The proper length of the stick is L_0 and we have $v = 0.8c$. Then:

(a) Parallel to its velocity, the apparent length of the stick is L given by:

$$L = L_0\sqrt{1 - v^2/c^2} = L_0\sqrt{1 - 0.64} = L_0\sqrt{0.36} = 0.6L_0$$

(b) Perpendicular to its velocity, the length is unchanged, therefore $L = L_0$.

(c) Let y_0 be the proper length component parallel to the velocity and let x_0 be the proper length component perpendicular to the velocity. Then $\sin 30° = \frac{y_0}{L_0}$ so that $y = \frac{1}{2}L_0$. Likewise $\cos 30° = \frac{x_0}{L_0}$ so that $x_0 = \frac{\sqrt{3}}{2}L_0$. Then the apparent lengths of the components are given by:

$$y = y_0\sqrt{1 - v^2/c^2} = 0.6(\frac{1}{2}L_0) = 0.3L_0$$

$$x = x_0 = \frac{\sqrt{3}}{2}L_0$$

Therefore the apparent length of the stick is:

$$\begin{aligned} L &= \sqrt{x^2 + y^2} \\ &= \sqrt{\frac{3}{4}L_0^2 + 0.09L_0^2} \\ &= \sqrt{0.84}L_0 \\ &\approx 0.92L_0 \end{aligned}$$

Section 7. *Relativistic Momementum*

7. *The mass of an electron is about* 9.11×10^{-31} *kg. Make a table showing an electron's momentum, both the correct relativistic momentum and the nonrelativistic form, at speeds with* $v = .1c, .5c, .9c, .99c$.

The classical, or nonrelativistic, momentum is $p_c = mv$. The relativistic momentum is $p_r = \gamma mv$ where

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

Then, for the various speeds v we obtain:

v	$p_c(\frac{kg \cdot m}{s})$	$p_r(\frac{kg \cdot m}{s})$
$0.1c$	2.73×10^{-23}	2.75×10^{-23}
$0.5c$	1.37×10^{-22}	1.58×10^{-22}
$0.9c$	2.46×10^{-22}	5.64×10^{-22}
$0.99c$	2.71×10^{-22}	1.92×10^{-21}

Section 8. *Mass and Energy*

8. *(book #29) At what speed does the kinetic energy of a particle equal its rest energy?*

We have that the kinetic energy $KE = (\gamma - 1)mc^2$ and the rest energy $E_0 = mc^2$. Therefore $KE = E_0$ when $\gamma = 2$. Therefore:

$$\begin{aligned}\gamma = 2 \Longrightarrow \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} &= 2 \\ \Longrightarrow 1 - \frac{v^2}{c^2} &= \frac{1}{4} \\ \Longrightarrow v^2 &= \frac{3}{4}c^2 \\ \Longrightarrow v &= \frac{\sqrt{3}}{2}c \approx 0.866c\end{aligned}$$

Therefore the kinetic energy of a particle equals its rest energy at the speed $\frac{\sqrt{3}}{2}c \approx 0.866c$

9. *How many joules of energy per kilogram of rest mass are needed to bring a spacecraft from rest to a speed of* $0.70c$*?*

For $m = 1$ kg and $v = 0.70c$, the kinetic energy required in Joules is given by:

$$\begin{aligned}KE &= (\gamma - 1)mc^2 \\ &= \left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] mc^2 \\ &\approx 0.40mc^2 \\ &\approx 3.6 \times 10^{16}\text{J}\end{aligned}$$

Therefore 3.6×10^{16} Joules of energy per kilogram of rest mass are needed to bring a spacecraft from rest to a speed of $0.70c$.

10. *(book #31) An electron has a kinetic energy of* 0.100 *MeV. Find its speed according to classical and relativistic mechanics.*

We must find v given $m_e = 9.11 \times 10^{-31}$ kg and $KE = 0.100 \times 10^6\text{eV} \times 1.602 \times 10^{-19}\text{J/eV} = 1.602 \times 10^{-14}\text{J}$.

Classically:

$$\begin{aligned}KE = \frac{1}{2}m_e v^2 \Longrightarrow v^2 &= \frac{2KE}{m_e} \\ \Longrightarrow v &= \sqrt{\frac{2KE}{m_e}} \\ \Longrightarrow v &\approx \sqrt{\frac{2(1.602 \times 10^{-14}\text{J})}{9.11 \times 10^{-31}\text{kg}}} \approx 1.88 \times 10^8\text{m/s}\end{aligned}$$

Therefore the speed of the electron according to classical mechanics is 1.88×10^8m/s.

Relativistically, and hence more realistically:

$$
\begin{aligned}
KE = \left[\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right] m_e c^2 &\Longrightarrow \frac{KE}{m_e c^2} = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1 \\
&\Longrightarrow \sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\frac{KE}{m_e c^2}+1} \\
&\Longrightarrow 1-\frac{v^2}{c^2} = \frac{1}{(\frac{KE}{m_e c^2}+1)^2} \\
&\Longrightarrow v^2 = c^2\left[1-\frac{1}{(\frac{KE}{m_e c^2}+1)^2}\right] \\
&\Longrightarrow v = c\sqrt{1-\frac{1}{(\frac{KE}{m_e c^2}+1)^2}} \\
&\Longrightarrow v \approx 1.64 \times 10^8 \text{m/s}
\end{aligned}
$$

Therefore the speed of the electron according to relativistic mechanics is 1.64×10^8m/s.

11. *(book #34) (a) The speed of a proton is increased from* $0.20c$ *to* $0.40c$*. By what factor does its kinetic energy increase? (b) The proton speed is again doubled, this time to* $0.80c$*. By what factor does its kinetic energy increase now?*

Let $v_0 = 0.20c$, $v_1 = 0.40c$, and $v_2 = 0.80c$. For any given v_x and mass m we have $KE_x = (\gamma_x - 1)mc^2$ where

$$\gamma_x = \frac{1}{\sqrt{1-\frac{v_x^2}{c^2}}}$$

Thus we have,

$$
\begin{aligned}
\gamma_0 &= \frac{1}{\sqrt{1-0.04}} \approx 1.0206 \\
\gamma_1 &= \frac{1}{\sqrt{1-0.16}} \approx 1.0911 \\
\gamma_2 &= \frac{1}{\sqrt{1-0.64}} \approx 1.6667
\end{aligned}
$$

Then, the factor by which the proton's kinetic energy increases as its speed increases from v_0 to v_1 is

(a)
$$\frac{KE_1}{KE_0} = \frac{\gamma_1 - 1}{\gamma_0 - 1} \approx 4.42$$

and the factor by which the proton's kinetic energy increases as its speed increases from v_1 to v_2 is

(b)
$$\frac{KE_2}{KE_1} = \frac{\gamma_2 - 1}{\gamma_1 - 1} \approx 7.32$$