

Modern Physics: Chapter 5 Homework

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1. (book #2) Which of the wave functions in Fig. 5.16 cannot have physical significance in the interval shown? Why not?

(b) has no value for some values of x .

(c) is not continuous.

(e) is not single-valued.

2. (book #3) Which of the following wave functions cannot be solutions of Schrödinger's equation for all values of x ? Why not? (a) $\psi = A \sec x$; (b) $\psi = A \tan x$; (c) $\psi = Ae^{x^2}$; (d) $\psi = Ae^{-x^2}$.

(a) $\psi = A \sec x$ is discontinuous since $\sec \frac{\pi}{2}$ is undefined.

(b) $\psi = A \tan x$ is discontinuous since $\tan \frac{\pi}{2}$ is undefined.

(c) $\psi = Ae^{x^2}$ is not normalizable since $\lim_{x \rightarrow \infty} (\psi) \neq 0$.

3. (book #7) As mentioned in Section 5.1, in order to give physically meaningful results in calculations a wave function and its partial derivatives must be finite, continuous, and single-valued, and in addition must be normalizable. Equation 5.9 gives the wave function of a particle moving freely (that is, with no forces acting on it) in the $+x$ direction as

$$\Psi = Ae^{-\frac{i}{\hbar}(Et - px)}$$

where E is the particle's total energy and p is its momentum. Does this wave function meet all the above requirements? If not, could a linear superposition of such wave functions meet these requirements? What is the significance of such a superposition of wave functions?

Ψ above does not meet all the requirements.

Ψ is not normalizable since $\lim_{x \rightarrow \infty} \Psi$ does not exist.

A linear superposition of these wave functions, however, could meet the requirements. For instance, the Fourier integral depicted in Figure 3.13 does so. Such a superposition would yield a wave group that could correspond to a real particle.

4. (book #9) Show that the expectation values $\langle px \rangle$ and $\langle xp \rangle$ are related by

$$\langle px \rangle - \langle xp \rangle = \frac{\hbar}{i}$$

This result is described by saying that p and x do not **commute** and it is intimately related to the uncertainty principle.

We have,

$$\begin{aligned} \langle px \rangle - \langle xp \rangle &= \left[\int_{-\infty}^{\infty} \Psi^* \left(\frac{\hbar}{i} \right) \frac{\partial}{\partial x} (x\Psi) dx \right] - \left[\int_{-\infty}^{\infty} \Psi^* x \left(\frac{\hbar}{i} \right) \frac{\partial}{\partial x} \Psi dx \right] \\ &= \left(\frac{\hbar}{i} \right) \left[\int_{-\infty}^{\infty} \Psi^* \left(x \frac{\partial \Psi}{\partial x} + \Psi \right) dx \right] - \left(\frac{\hbar}{i} \right) \left[\int_{-\infty}^{\infty} \Psi^* x \frac{\partial \Psi}{\partial x} dx \right] \\ &= \left(\frac{\hbar}{i} \right) \left[\left(\int_{-\infty}^{\infty} \Psi^* x \frac{\partial \Psi}{\partial x} dx \right) + \left(\int_{-\infty}^{\infty} \Psi^* \Psi dx \right) - \left(\int_{-\infty}^{\infty} \Psi^* x \frac{\partial \Psi}{\partial x} dx \right) \right] \\ &= \left(\frac{\hbar}{i} \right) \int_{-\infty}^{\infty} \Psi^* \Psi dx \\ &= \left(\frac{\hbar}{i} \right) \int_{-\infty}^{\infty} |\Psi|^2 dx \\ &= \left(\frac{\hbar}{i} \right) \end{aligned}$$

Q.E.D.

5. (book #10) An eigenfunction of the operator $\frac{d^2}{dx^2}$ is $\sin nx$, where $n = 1, 2, 3, \dots$. Find the corresponding eigenvalues.

Let z_n be the eigenvalue for any given n . Then,

$$\frac{d^2}{dx^2} \sin nx = z_n \sin nx$$

And we see that

$$\frac{d^2}{dx^2} \sin nx = \frac{d}{dx} (n \cos nx) = -n^2 \sin nx$$

so that $z_n = \boxed{-n^2}$.

6. (book #14) In Section 5.8 a box was considered that extends from $x = 0$ to $x = L$. Suppose the box instead extends from $x = x_0$ to $x = x_0 + L$, where $x_0 \neq 0$. Would the expression for the wave functions of a particle in this box be any different from those in the box that extends from $x = 0$ to $x = L$? Would the energy levels be different?

We could simply express the wave function of a particle in the box that extends from $x = x_0$ to $x = x_0 + L$ as

$$\psi_n = A \sin \frac{\sqrt{2mE_n}}{\hbar} (x - x_0)$$

The derivation of the energy levels then remains the same, yielding

$$E_n = \frac{n^2 \pi^2 \hbar^2}{2mL^2}, \quad n = 1, 2, 3, \dots$$

7. (book #19) Find the probability that a particle in a box L wide can be found between $x = 0$ and $x = L/n$ when it is in the n^{th} state.

The probability of finding the particle between any x_1 and x_2 when the particle is in the n^{th} state is

$$P_{x_1, x_2} = \int_{x_1}^{x_2} |\psi_n|^2 dx, \quad \text{where } \psi_n = \sqrt{\frac{2}{L}} \sin \frac{n\pi x}{L}$$

Therefore,

$$\begin{aligned} P_{0, \frac{L}{n}} &= \int_0^{\frac{L}{n}} \frac{2}{L} \sin^2 \frac{n\pi x}{L} dx \\ &= \frac{2}{L} \int_0^{\frac{L}{n}} \frac{1}{2} \left(1 - \cos \frac{2n\pi x}{L} \right) dx \\ &= \frac{1}{L} \left[x - \frac{L}{2n\pi} \sin \frac{2n\pi x}{L} \right]_0^{\frac{L}{n}} \\ &= \frac{1}{L} \left[\left(\frac{L}{n} - \frac{L}{2n\pi} \sin 2\pi \right) - \left(0 - \frac{L}{2n\pi} \sin 0 \right) \right] \\ &= \boxed{\frac{1}{n}} \end{aligned}$$

Hence the probability of finding the particle between $x = 0$ and $x = L/n$ is $1/n$.

8. (book #21) A particle is in a cubic box with infinitely hard walls whose edges are L long. The wave functions of the particle are given by

$$\psi = A \sin \frac{n_x \pi x}{L} \sin \frac{n_y \pi y}{L} \sin \frac{n_z \pi z}{L}, \quad n_x, n_y, n_z \in N$$

Find the value of the normalization constant A .

We have,

$$1 = \int_0^L \int_0^L \int_0^L A^2 \sin^2 \frac{n_x \pi x}{L} \sin^2 \frac{n_y \pi y}{L} \sin^2 \frac{n_z \pi z}{L} dz dy dx$$

Then since

$$\begin{aligned} \int_0^L \sin^2 \frac{n_u \pi u}{L} du &= \int_0^L \frac{1}{2} \left(1 - \cos \frac{2n_u \pi u}{L} \right) du \\ &= \left[\frac{u}{2} - \frac{L}{2n\pi} \sin \frac{2n_u \pi u}{L} \right]_0^L \\ &= \frac{L}{2}, \end{aligned}$$

consequently,

$$\begin{aligned}
 1 &= A^2 \int_0^L \int_0^L \sin^2 \frac{n_x \pi x}{L} \sin^2 \frac{n_y \pi y}{L} \left(\frac{L}{2}\right) dy dx \\
 &= A^2 \int_0^L \sin^2 \frac{n_x \pi x}{L} \left(\frac{L}{2}\right) \left(\frac{L}{2}\right) dx \\
 &= A^2 \left(\frac{L}{2}\right) \left(\frac{L}{2}\right) \left(\frac{L}{2}\right)
 \end{aligned}$$

Therefore $A^2 = \left(\frac{2}{L}\right)^3$ so that $A = \boxed{\left(\frac{2}{L}\right)^{3/2}}$

9. (book #25) A beam of electrons is incident on a barrier 6.00eV high and 0.200nm wide. Use

$$T = e^{-2k_2L} \quad \text{where} \quad k_2 = \frac{\sqrt{2m(U - E)}}{\hbar}$$

to find the energy they should have if 1.00 percent of them are to get through the barrier.

We have $U = 6.00\text{eV}$ and $L = 0.200 \times 10^{-9}\text{m}$ and $T = 0.0100$. Also $m = m_e = 9.109 \times 10^{-31}\text{kg}$ and $\hbar = 1.054 \times 10^{-34}\text{J} \cdot \text{s}$.

Now,

$$\begin{aligned}
 k_2 &= \frac{\sqrt{2m(U - E)}}{\hbar} \implies (k_2\hbar)^2 = 2m(U - E) \\
 &\implies \frac{(k_2\hbar)^2}{2m} = U - E \\
 &\implies E = U - \frac{(k_2\hbar)^2}{2m}
 \end{aligned}$$

And

$$\begin{aligned}
 T &= e^{-2k_2L} \implies \ln T = -2k_2L \\
 &\implies k_2 = -\frac{\ln T}{2L}
 \end{aligned}$$

Therefore

$$E = U - \frac{(\ln T)^2 \hbar^2}{4L^2 \cdot 2m} = U - \frac{(\ln T)^2 \hbar^2}{8mL^2}$$

so that

$$\begin{aligned}
 E &= (6.00\text{eV})(1.602 \times 10^{-19}\text{J/eV}) - \frac{(\ln 0.0100)^2 (1.054 \times 10^{-34}\text{J} \cdot \text{s})^2}{8(9.109 \times 10^{-31}\text{kg})(0.200 \times 10^{-9}\text{m})^2} \\
 &= 1.53 \times 10^{-19}\text{J}
 \end{aligned}$$

Therefore electrons in the beam should have an energy of at least $\boxed{1.53 \times 10^{-19}\text{J}}$ if 1.00 percent of them are to get through the barrier.

10. (book #29) Show that for the $n = 0$ state of a harmonic oscillator whose classical amplitude of motion is A , $y = 1$ at $x = A$, where

$$y = \left(\frac{1}{\hbar} \sqrt{km} \right)^{1/2} x.$$

We have that when $n = 0$, $E = \frac{1}{2}h\nu$ where $\nu = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$ and $h = 2\pi\hbar$.
When $x = A$, $U = E$ so that

$$\begin{aligned} U = E &\implies \frac{1}{2}kx^2 = \frac{1}{2}h\nu \\ &\implies x^2 = \frac{h\nu}{k} = \frac{\hbar \sqrt{\frac{k}{m}}}{k} \\ &\implies x^2 = \frac{\hbar}{\sqrt{km}} \end{aligned}$$

Therefore

$$y = \left(\frac{\sqrt{km}}{\hbar} \right)^{1/2} \left(\frac{\hbar}{\sqrt{km}} \right)^{1/2} = 1$$

Q.E.D.

11. (book #30) Find the probability density $|\psi_0|^2 dx$ at $x = 0$ and at $x = \pm A$ of a harmonic oscillator in its $n = 0$ state.

We have,

$$\psi_n = \left(\frac{2m\nu}{\hbar} \right)^{1/4} (2^n n!)^{-1/2} H_n(y) e^{-y^2/2} \quad \text{with } n = 0 \implies H_n(y) = 1$$

Therefore

$$\psi_n = \left(\frac{2m\nu}{\hbar} \right)^{1/4} e^{-y^2/2}$$

Now, when $x = 0$, $y = 0$ so that

$$\boxed{|\psi_0|^2 dx = \left(\frac{2m\nu}{\hbar} \right)^{1/2} dx}$$

And when $x = \pm A$, $y = 1$ (as is easily seen by the example of problem #10 above) so that

$$\boxed{|\psi_0|^2 dx = \left(\frac{2m\nu}{\hbar} \right)^{1/2} e^{-1} dx}$$