Algebra

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4.8 Proposition. *Let H be a subgroup of a group G and let G act on the set S of all left cosets of H in G by left translation. Then the kernel of the induced homomorphism $G \to A(S)$, where $A(S)$ is the set of all permutations of S, is contained in H.*

Suppose $H < G$ and $S = \{xH : x \in G\}$. Let $\theta_g : S \to S$ be defined by $\theta_g(xH) = gxH$ for each $g \in G$. Then $(\theta_g, xH) \mapsto \theta_g(xH)$ is an action by left translation of G on S, the set of all left cosets of H in G. Therefore by Theorem 4.5, this action induces a homomorphism $\phi : G \to A(S)$ given by $g \mapsto \theta_g$ for all $g \in G$. We need to show $\operatorname{Ker}\phi \subseteq H$. To do this we will show that $g \in \operatorname{Ker}\phi \implies g \in H$.

We have,

$$\begin{aligned} g \in \operatorname{Ker}\phi &\implies \theta_g = 1_S \\ &\implies \theta_g(xH) = xH \quad \forall xH \in S \\ &\implies gxH = xH \quad \forall xH \in S \\ &\implies (gx)x^{-1} \in H \quad \forall x \in G \\ &\implies g(xx^{-1}) \in H \quad \forall x \in G \\ &\implies g \in H \end{aligned}$$

Therefore $g \in \operatorname{Ker}\phi \implies g \in H$.

Therefore $\operatorname{Ker}\phi \subseteq H$. **Q.E.D.**

4.10 Corollary. *If H is a subgroup of a finite group G of index p, where p is the smallest prime dividing the order of G, then H is normal in G.*

Suppose $H < G$ such that $[H : G] = p$ for prime p, p divides $|G|$, and q does not divide $|G|$ for all primes $q < p$. We need to show that $H \triangleleft G$.

Let S be the set of all left cosets of H in G and let G act on the set S by left translation. Let K be the kernel of the induced homomorphism $G \to A(S)$ which we know exists by Theorem 4.5. Then by Proposition 4.8, $K \subseteq H$. Furthermore, by Theorem I.5.5, $K \triangleleft G$. We will show that $H = K$ and hence $H \triangleleft G$.

By the First Isomorphism Theorem, G/K is isomorphic to a subgroup of $A(S)$. $|S| = p$ since $|G : H| = p$ so that $A(S) \cong S_p$, therefore G/K is isomorphic to a subgroup of S_p. Therefore by Lagrange, $|G/K|$ divides $|S_p| = p!$. But $G/K < G$ so that Lagrange also implies that $|G/K|$ divides $|G|$. Therefore since no prime less than p divides $|G|$, and every divisor of $p!$ is divisible by a prime less than or equal to p, it must be that $|G/K| \in \{p, 1\}$. Now $K \subseteq H$ and $K < G$ implies that $K < H$ so that

$$\begin{aligned}
|G/K| &= [G : K] \\
&= [G : H][H : K] \\
&= p[H : K] \\
&\neq 1
\end{aligned}$$

Therefore

$$\begin{aligned}
|G/K| = p &\Longrightarrow p[H : K] = p \\
&\Longrightarrow [H : K] = 1 \\
&\Longrightarrow H = K \\
&\Longrightarrow H \triangleleft G \quad \text{since } K \triangleleft G
\end{aligned}$$

Q.E.D.