Modern Physics: Chapter 6 Homework

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1. *(book #2) Show that*

$$\Theta_{20}(\theta) = \frac{\sqrt{10}}{4}(3\cos^2\theta - 1)$$

is a solution of Equation (6.13) and that it is normalized.

We need to show that

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) + \left[l(l+1) - \frac{m_l^2}{\sin^2\theta}\right]\Theta = 0 \tag{1}$$

for $l = 2$, $m_l = 0$, and for all values of θ.

We have $\frac{d\Theta}{d\theta} = \frac{-3\sqrt{10}}{2}\cos\theta\sin\theta$, so that the expression on the left hand side of (1) above is

$$\begin{aligned}
&\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\frac{-3\sqrt{10}}{2}\cos\theta\sin^2\theta\right) + \left[6\frac{\sqrt{10}}{4}\left(3\cos^2\theta - 1\right)\right] \\
&\qquad = \frac{1}{\sin\theta}\left(\frac{-3\sqrt{10}}{2}\right)\left(2\cos^2\theta\sin\theta - \sin^3\theta\right) + \frac{3\sqrt{10}}{2}\left(3\cos^2\theta - 1\right) \\
&\qquad = \frac{-3\sqrt{10}}{2}\left(2\cos^2\theta - \sin^2\theta\right) + \frac{3\sqrt{10}}{2}\left(3\cos^2\theta - 1\right) \\
&\qquad = 0 \quad \text{since } \sin^2\theta + \cos^2\theta = 1 \Longrightarrow \sin^2\theta = 1 - \cos^2\theta
\end{aligned}$$

To show that Θ is normalized, we will show that $\int_0^\pi |\Theta|^2 \sin\theta\, d\theta = 1$.
We have,

$$\begin{aligned}
\int_0^\pi |\Theta|^2 \sin\theta\, d\theta &= \int_0^\pi \frac{10}{16}\left(3\cos^2\theta - 1\right)^2 \sin\theta\, d\theta \\
&= -\frac{5}{8}\int_1^{-1} (3u^2 - 1)^2\, du \quad \text{with } u = \cos\theta, \quad du = -\sin\theta\, d\theta \\
&= -\frac{5}{8}\int_1^{-1} (9u^4 - 6u^2 + 1)\, du \\
&= -\frac{5}{8}\left[\frac{9u^5}{5} - \frac{6u^3}{3} + u\right]_1^{-1} \\
&= -\frac{5}{8}\left(-\frac{9}{5} + 2 - 1 - \frac{9}{5} + 2 - 1\right) \\
&= -\frac{5}{8}\left(-\frac{8}{5}\right) = 1
\end{aligned}$$

Q.E.D.

We will use the following Lemma in problems #2 and #8:

Lemma 1. $\int_0^\infty r^k e^{-pr}\,dr = \left(\frac{1}{p}\right)^{k+1} k!$ *for* $k \in N$, $p > 0$

Let $u = r^k$, $dv = e^{-pr}\,dr$ and hence $du = kr^{k-1}\,dr$, $v = \frac{-1}{p}e^{-pr}$. Then,

$$\begin{aligned}
\int_0^\infty r^k e^{-pr}\,dr &= \int_0^\infty u\,dv \\
&= \left[r^k\left(-\frac{1}{p}e^{-pr}\right)\right]_0^\infty + \int_0^\infty \frac{1}{p}e^{-pr}\cdot kr^{k-1}\,dr \\
&= \frac{k}{p}\int_0^\infty r^{k-1}e^{-pr}\,dr \\
&= \left(\frac{1}{p}\right)^k k! \int_0^\infty e^{-pr}\,dr \\
&= -\left(\frac{1}{p}\right)^{k+1} k! \left[e^{-pr}\right]_0^\infty \\
&= \left(\frac{1}{p}\right)^{k+1} k!
\end{aligned}$$

Q.E.D.

2. *(book #4) Show that*

$$R_{21}(r) = \frac{1}{2\sqrt{6}a_0^{3/2}}\frac{r}{a_0}e^{-r/2a_0}$$

is a solution of Equation (6.14) and that it is normalized.

We need to show that

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r} + E\right) - \frac{l(l+1)}{r^2}\right]R = 0$$

for $n = 2$, $l = 1$, and for all values of r.

We have

$$\begin{aligned}
\frac{dR}{dr} &= \frac{1}{2\sqrt{6}a_0^{5/2}}\frac{d}{dr}\left(re^{-r/2a_0}\right) \\
&= \frac{1}{2\sqrt{6}a_0^{5/2}}\left(\frac{-r}{2a_0}e^{-r/2a_0} + e^{-r/2a_0}\right) \\
&= \frac{e^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}\left(\frac{-r}{2a_0} + 1\right)
\end{aligned}$$

Then

$$\begin{aligned}
\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= \frac{d}{dr}\left[\frac{r^2e^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}\left(\frac{-r}{2a_0}+1\right)\right] \\
&= \frac{1}{2\sqrt{6}a_0^{5/2}}\frac{d}{dr}\left[r^2e^{-r/2a_0}\left(\frac{-r}{2a_0}+1\right)\right] \\
&= \frac{1}{2\sqrt{6}a_0^{5/2}}\left[\frac{-r^2e^{-r/2a_0}}{2a_0}+\left(\frac{-r}{2a_0}+1\right)\left(\frac{-r^2e^{-r/2a_0}}{2a_0}+2re^{-r/2a_0}\right)\right] \\
&= \frac{re^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}\left[\frac{-r}{2a_0}+\left(\frac{-r}{2a_0}+1\right)\left(\frac{-r}{2a_0}+2\right)\right]
\end{aligned}$$

so that

$$\begin{aligned}
\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) &= \frac{e^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}\left[\frac{-1}{2a_0}+\frac{r}{4a_0^2}-\frac{1}{a_0}-\frac{1}{2a_0}+\frac{2}{r}\right] \\
&= \frac{e^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}\left(\frac{-2}{a_0}+\frac{r}{4a_0^2}+\frac{2}{r}\right)
\end{aligned}$$

Meanwhile,

$$\left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r}+E\right)-\frac{l(l+1)}{r^2}\right]R = \left[\frac{2m}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r}+E\right)-\frac{2}{r^2}\right]\frac{re^{-r/2a_0}}{2\sqrt{6}a_0^{5/2}}$$

so that it suffices to show that

$$-\frac{2}{a_0}+\frac{r}{4a_0^2}+\frac{2}{r}+\frac{2mr}{\hbar^2}\left(\frac{e^2}{4\pi\epsilon_0 r}+E\right)-\frac{2}{r}=0$$

On the left hand side we have

$$\frac{-2}{a_0}+\frac{r}{4a_0^2}+\frac{2mre^2}{4\hbar\pi\epsilon_0 r}+\frac{2mrE}{\hbar^2}$$

Then substituting for $a_0 = \dfrac{4\pi\hbar^2\epsilon_0}{me^2}$ we have

$$\begin{aligned}
\frac{-2me^2}{4\pi\hbar^2\epsilon_0}+\frac{rm^2e^4}{4\cdot 16\pi^2\hbar^4\epsilon_0^2}+\frac{2me^2}{4\hbar^2\pi\epsilon_0}+\frac{2mrE}{\hbar^2} &= \frac{rm^2e^4}{64\pi^2\hbar^4\epsilon_0^2}+\frac{2mrE}{\hbar^2} \\
&= \frac{2rm}{\hbar^2}\left(\frac{me^4}{32\pi^2\hbar^2\epsilon_0^2}\left(\frac{1}{2^2}\right)+E\right)
\end{aligned}$$

which is equal to zero precisely when

$$E = -\frac{me^4}{32\pi^2\hbar^2\epsilon_0^2}\left(\frac{1}{n^2}\right)$$

which corresponds to the energy levels of the hydrogen atom. Therefore R is a solution of Equation (6.14).

To show that R is normalized, we will show that

$$\int_0^\infty r^2 |R|^2 \, dr = 1$$

.

We have

$$\begin{aligned}
\int_0^\infty r^2 |R|^2 \, dr &= \int_0^\infty \frac{r^4}{24a_0^5} e^{-r/a_0} \, dr \\
&= \frac{1}{24a_0^5} \int_0^\infty r^4 e^{-r/a_0} \, dr \\
&= \frac{1}{24a_0^5} (a_0)^5 \, 4! \quad \text{by Lemma 1} \\
&= 1
\end{aligned}$$

Q.E.D.

3. *(book #9) Under what circumstances, if any, is L_z equal to L?*

Only when $L = 0$ can L_z equal L. Otherwise $|L| > |L_z|$.

4. *(Book #11) What are the possible values of the magnetic quantum number m_l of an atomic electron whose orbital quantum number is $l = 4$?*

Since $l \geq |m_l|$, we have $\boxed{m_l \in \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}}$.

5. *(Book #12) List the sets of quantum numbers possible for an $n = 4$ hydrogen atom.*

We may have $l \in \{0, 1, 2, 3\}$. Then:
When $l = 0$, $m_l = 0$.
When $l = 1$, $m_l \in \{-1, 0, 1\}$.
When $l = 2$, $m_l \in \{-2, -1, 0, 1, 2\}$.
When $l = 3$, $m_l \in \{-3, -2, -1, 0, 1, 2, 3\}$.
These comprise all quantum numbers possible for an $n = 4$ hydrogen atom.

6. *(Book #17) Find the most probable value of r for a 3d electron in a hydrogen atom.*

For a $3d$ electron, we have

$$R_{32}(r) = \frac{4r^2 e^{-r/3a_0}}{81\sqrt{30}a_0^{7/2}}$$

Then the probability of finding the electron at radius r is given by

$$P(r) = r^2 |R|^2 = \frac{4^2 r^6 e^{-2r/3a_0}}{81^2 \cdot 30a_0^7}$$

.

We need to find r for which $P(r)$ is at maximum. Hence we have

$$\frac{d}{dr}P(r) = \frac{4^2}{81^2 \cdot 30a_0^7}\frac{d}{dr}\left(r^6 e^{-2r/3a_0}\right) = 0$$
$$\Longrightarrow \frac{-2r^6 e^{-2r/3a_0}}{3a_0} + 6r^5 e^{-2r/3a_0} = 0$$

But $r = 0$ is a minimum since $P(0) = 0$. Therefore

$$\frac{-2r}{3a_0} + 6 = 0$$
$$\Longrightarrow 6 = \frac{2}{3a_0}r$$
$$\Longrightarrow r = 9a_0$$

Therefore $\boxed{9a_0}$ is the most probable radius for a $3d$ electron in a hydrogen atom.

7. *(Book #19) How much more likely is the electron in a ground-state hydrogen atom to be at the distance a_0 from the nucleus than at the distance $2a_0$?*

For an electron in the ground state,

$$R = \frac{2}{a_0^{3/2}}e^{-r/a_0}$$

Then letting P_1 be the probability that the electron will be at $r_1 = a_0$ and letting P_2 be the probability that the electron will be at $r_2 = 2a_0$, we need to find P_1/P_2.

We have

$$\frac{P_1}{P_2} = \frac{r_1^2|R_1|^2}{r_2^2|R_2|^2} = \frac{(a_0)^2 e^{-2a_0/a_0}}{(2a_0)^2 e^{-4a_0/a_0}} = \frac{1e^{-2}}{4e^{-4}} = \frac{1}{4}e^2 = 1.85$$

Therefore it is $\boxed{1.85}$ times as likely that the electron will be found at radius a_0 than at $2a_0$.

8. *(Book #20) Verify that the average value of r for a 1s electron in a hydrogen atom is $1.5a_0$ by calculating the expectation value $\langle r\rangle = \int r|\psi|^2\,dV$.*

We have

$$\begin{aligned}
\langle r\rangle &= \int_0^\infty r|R_{10}|^2 r^2\,dr \\
&= \int_0^\infty r^3 \left(\frac{2}{a_0^{3/2}} e^{-r/a_0}\right)^2 dr \\
&= \frac{4}{a_0^3}\int_0^\infty r^3 e^{-2r/a_0}\,dr \\
&= \frac{4}{a_0^3}\left(\frac{a_0}{2}\right)^4 3! \quad \text{by Lemma 1} \\
&= \frac{4a_0}{2^4}(3\cdot 2) \\
&= \frac{3}{2}a_0
\end{aligned}$$

Therefore the average value of r for a $1s$ electron in the hydrogen atom is $1.5a_0$. **Q.E.D.**

9. *(Book #24) A hydrogen atom is in the 4p state. To what state or states can it go by radiating a photon in an allowed transition?*

By the selection rule $\Delta l = \pm 1$, the electron beginning in the $n = 4$, $l = 1$ state may go to any of the following states by radiating a photon: $\boxed{3s, 3d, 2s, 1s}$.