Answers to Starred Problems
Sheet 4
MATH 3314
Dr. Baker
David Joseph Stith

Let f_n denote the Fibonacci number given by:

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for } n > 1$$

Theorem 6c. $f_{n+1}f_{n-1} - f_n^2 = (-1)^n$ *for all positive integers* n.

Let $P(n)$ denote the proposition to be proven. We see that $P(1)$ is equivalent to the following:

$$\begin{aligned} f_2 f_0 - f_1^2 &= (-1)^1 \\ 1 \cdot 0 - 1^2 &= -1 \\ -1 &= -1 \end{aligned}$$

which is clearly true.

Now we shall show that $P(k) \to P(k+1)$. $P(k+1)$ is equivalent to the following:

$$\begin{aligned} f_{k+2}f_k - f_{k+1}^2 &= (-1)^{k+1} \\ f_{k+2}f_k - f_{k+1}^2 &= -(-1)^k \\ f_{k+1}^2 - f_{k+2}f_k &= (-1)^k \end{aligned}$$

And by definition we have $f_{k+2} = f_{k+1} + f_k$ so that $P(k+1)$ is furthermore equivalent to:

$$\begin{aligned} f_{k+1}^2 - (f_{k+1} + f_k)f_k &= (-1)^k \\ f_{k+1}(f_{k+1} - f_k) - f_k^2 &= (-1)^k \end{aligned}$$

Now, $f_{k+1} - f_k = f_{k-1}$ follows from the definition, hence $P(k+1)$ is furthermore equivalent to:

$$f_{k+1}f_{k-1} - f_k^2 = (-1)^k$$

which is equivalent to $P(k)$. Therefore $P(k) \to P(k+1)$.

This combined with the truth of $P(1)$ shown previously shows that by mathematical induction $P(n)$ is true for all positive integers n. **QED**

Let f_n denote the Fibonacci number given by:

$$f_0 = 0$$
$$f_1 = 1$$
$$f_n = f_{n-1} + f_{n-2} \text{ for } n > 1$$

Theorem 6d. *$f_1 f_2 + f_2 f_3 + \cdots + f_{2n-1} f_{2n} = f_{2n}^2$ for all positive integers n.*

Let $P(n)$ denote the proposition to be proven, that is:

$$\sum_{i=1}^{2n-1} f_i f_{i+1} = f_{2n}^2$$

We see that $P(1)$ is true, since the following are equivalent:

$$\sum_{i=1}^{1} f_i f_{i+1} = f_2^2$$
$$f_1 f_2 = f_2^2$$
$$1 \cdot 1 = 1^2$$

Now we shall show that $P(k) \to P(k+1)$. First we note that $P(k+1)$ is equivalent to the following:

$$\sum_{i=1}^{2(k+1)-1} f_i f_{i+1} = f_{2(k+1)}^2$$
$$\left(\sum_{i=1}^{2k-1} f_i f_{i+1}\right) + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2(k+1)}^2$$

so that assuming $P(k)$ were true would allow us to replace the sum, so that $P(k+1)$ would be equivalent to the following:

$$f_{2k}^2 + f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2(k+1)}^2$$
$$f_{2k} f_{2k+1} + f_{2k+1} f_{2k+2} = f_{2k+2}^2 - f_{2k}^2$$
$$f_{2k+1}(f_{2k} + f_{2k+2}) = f_{2k+2}^2 - f_{2k}^2$$

and since $f_{2k+1} = f_{2k+2} - f_{2k}$ follows from the definition, $P(k+1)$ is furthermore equivalent to:

$$(f_{2k+2} - f_{2k})(f_{2k+2} + f_{2k}) = f_{2k+2}^2 - f_{2k}^2$$

which is clearly true.

Therefore since $P(k) \to P(k+1)$ and $P(1)$ is true, the principle of mathematical induction allows us to conclude that $P(n)$ is true for all positive integers n. **QED**

Problem 12b. *Find the value of Ackerman's function at $A(3,3)$.*

Let the notation $A_m(n)$ represent $A(m,n)$. Then, restricting ourselves to when $n \geq 1$, we have

$$\begin{aligned} A_m(1) &= 2 \\ A_0(n) &= 2n \\ A_m(n) &= (A_{m-1} \circ A_m)(n-1) \text{ for } n > 1 \text{ and } m > 0 \end{aligned}$$

Therefore, by $n-1$ recursive applications of the definition for $A_m(n)$, we have

$$\begin{aligned} A_m(n) &= (A_{m-1} \circ A_{m-1} \circ \cdots \circ A_{m-1} \circ A_m)(1) \\ &= (A_{m-1} \circ A_{m-1} \circ \cdots \circ A_{m-1})(2) \end{aligned}$$

where A_{m-1} appears $n-1$ times.

Consequently, $A_1(n)$ equals the multiplication of n occurences of 2, or simply 2^n. By using the notation $2 \uparrow n$ to represent multiplication of n occurences of 2, we shall then be able to write $A_2(n)$ as $2 \uparrow\uparrow n$ meaning $2 \uparrow (2 \uparrow \ldots (2 \uparrow 2) \ldots)$ in which the number 2 has appeared n times. Taking this a step further, we see that $A_m(n)$ can be written as $2 \uparrow\uparrow \cdots \uparrow n$ in which there are m occurences of the symbol $\uparrow$.

Now, we evaluate $A_3(3)$ as follows:

$$\begin{aligned} A_3(3) &= 2 \uparrow\uparrow\uparrow 3 \\ &= 2 \uparrow\uparrow (2 \uparrow\uparrow 2) \\ &= 2 \uparrow\uparrow (2 \uparrow 2) \\ &= 2 \uparrow\uparrow 4 \\ &= 2 \uparrow (2 \uparrow (2 \uparrow 2)) \\ &= 2 \uparrow (2 \uparrow 4)) \\ &= 2 \uparrow 16 \\ &= 65536 \end{aligned}$$

Problem 12c. *Find the value of Ackerman's function at $A(3,4)$.*

$$\begin{aligned} A_3(4) &= 2 \uparrow\uparrow\uparrow 4 \\ &= 2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow\uparrow 2)) \\ &= 2 \uparrow\uparrow (2 \uparrow\uparrow (2 \uparrow 2)) \\ &= 2 \uparrow\uparrow (2 \uparrow\uparrow 4) \\ &= 2 \uparrow\uparrow (2 \uparrow (2 \uparrow (2 \uparrow 2))) \\ &= 2 \uparrow\uparrow (2 \uparrow (2 \uparrow 4)) \\ &= 2 \uparrow\uparrow (2 \uparrow 16) \\ &= 2 \uparrow\uparrow 65536 \end{aligned}$$

It is fruitless to expand this last result, which is equivalent to the expression $2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}$ in which the numeral 2 appears 65536 times.

Answers to Starred Problems
Sheet 5
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Let (x_i, y_i), $i = 1, 2, 3, 4, 5$ be five distinct points with integer coefficients in the xy plane.

Theorem 9. *There must be two of these points which have the property that the line interval joining them has a midpoint which has integer coefficients.*

Any two pairs (x_a, y_a) and (x_b, y_b) have a midpoint with integer coefficients if and only if

$$x_a - x_b = 0 \pmod 2 \text{ and } y_a - y_b = 0 \pmod 2$$

which is equivalent to the following:

$$x_a = x_b \pmod 2 \text{ and } y_a = y_b \pmod 2$$

Now, we can classify all pairs (x_i, y_i) among four distinct and mutually exclusive groups:

$$\begin{cases} x_i = 0 \pmod 2 \text{ and } y_i = 0 \pmod 2 \\ x_i = 1 \pmod 2 \text{ and } y_i = 0 \pmod 2 \\ x_i = 0 \pmod 2 \text{ and } y_i = 1 \pmod 2 \\ x_i = 1 \pmod 2 \text{ and } y_i = 1 \pmod 2 \end{cases}$$

so that any two members of the same group will have a midpoint with integer coefficients.

Therefore, by choosing 5 pairs we insure that, according to the pigeonhole principle, at least 2 pairs will belong to the same group (since 5 is one more than the number of groups) and hence will have a midpoint with integer coefficients. **QED**