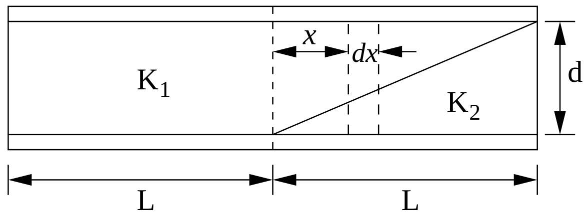
Robert Ramsey **PHYS1444 Quiz** David Joseph Stith

Problem. *Determine a formula for the equivalent capacitance for the parallel plate capacitor shown which has dielectrics κ_1 and κ_2 inserted, total length $2L$, a given plate area of A, and plate separation d.*



$$
\begin{aligned}
C_1 &= \kappa_1 \epsilon_0 \frac{\frac{1}{2}A}{d} \\
C_2 &= \int_0^L \left(\frac{d(\frac{L-x}{L})}{\kappa_1 \epsilon_0 \frac{A\,dx}{2L}} + \frac{d(\frac{x}{L})}{\kappa_2 \epsilon_0 \frac{A\,dx}{2L}} \right)^{-1} \\
&= \int_0^L \left(\frac{2d \cdot (L-x)}{\kappa_1 \epsilon_0 A\,dx} + \frac{2d \cdot x}{\kappa_2 \epsilon_0 A\,dx} \right)^{-1} \\
&= \int_0^L \left(\frac{2d \cdot \kappa_2 (L-x) + 2d \cdot \kappa_1 x}{\kappa_1 \kappa_2 \epsilon_0 A\,dx} \right)^{-1} \\
&= \int_0^L \frac{\kappa_1 \kappa_2 \epsilon_0 A\,dx}{2d[\kappa_2 L + x(\kappa_1 - \kappa_2)]} \\
&= \frac{\kappa_1 \kappa_2 \epsilon_0 A}{2d} \int_0^L \frac{dx}{(\kappa_1 - \kappa_2)x + \kappa_2 L} \\
&= \frac{\kappa_1 \kappa_2 \epsilon_0 A}{2d} \left[\frac{1}{\kappa_1 - \kappa_2} \ln\big((\kappa_1 - \kappa_2)x + \kappa_2 L\big) \right]_0^L \\
&= \frac{\kappa_1 \kappa_2 \epsilon_0 A}{2d(\kappa_1 - \kappa_2)} \Big[\ln\big((\kappa_1 - \kappa_2)L + \kappa_2 L\big) - \ln(\kappa_2 L) \Big] \\
&= \frac{\kappa_1 \kappa_2 \epsilon_0 A}{2d(\kappa_1 - \kappa_2)} \ln\left(\frac{\kappa_1}{\kappa_2}\right) \\
C_{eq} &= C_1 + C_2 \\
&= \kappa_1 \epsilon_0 \frac{A}{2d} + \frac{\kappa_1 \kappa_2 \epsilon_0 A}{2d(\kappa_1 - \kappa_2)} \ln\left(\frac{\kappa_1}{\kappa_2}\right)
\end{aligned}
$$

Therefore, when $\kappa_1 \neq \kappa_2$,

$$
\boxed{C_{eq} = \frac{\epsilon_0 A}{2d} \left(\kappa_1 + \frac{\kappa_1 \kappa_2 \ln(\frac{\kappa_1}{\kappa_2})}{\kappa_1 - \kappa_2} \right)}
$$

But when $\kappa_2 = \kappa_1$ we should have simply $\frac{\kappa_2 \epsilon_0 A}{d}$. We verify that this is consistent with our formula as follows:

$$
C_{\kappa_2 = \kappa_1} = \lim_{\kappa_1 \to \kappa_2} \frac{\epsilon_0 A}{2d} \left(\kappa_1 + \frac{\kappa_1 \kappa_2 \ln(\frac{\kappa_1}{\kappa_2})}{\kappa_1 - \kappa_2} \right) = \frac{\epsilon_0 A}{2d} \left(\kappa_2 + \kappa_2 \lim_{\kappa_1 \to \kappa_2} \frac{\kappa_1 \ln(\frac{\kappa_1}{\kappa_2})}{\kappa_1 - \kappa_2} \right)
$$

Then, since the limit produces the indeterminate form $\frac{0}{0}$, we apply l'Hôpital's rule, taking the limit of the numerator differentiated with respect to κ_1 over the denominator differentiated with respect to κ_1:

$$
\begin{aligned}
C_{\kappa_2 = \kappa_1} &= \frac{\epsilon_0 A}{2d} \left(\kappa_2 + \kappa_2 \lim_{\kappa_1 \to \kappa_2} \frac{\kappa_1 (\frac{\kappa_2}{\kappa_1})(\frac{1}{\kappa_2}) + \ln \frac{\kappa_1}{\kappa_2}}{1} \right) \\
&= \frac{\epsilon_0 A}{2d} \left(\kappa_2 + \kappa_2 + \lim_{\kappa_1 \to \kappa_2} \ln \frac{\kappa_1}{\kappa_2} \right) \\
&= \frac{\kappa_2 \epsilon_0 A}{d}
\end{aligned}
$$