Calculus Exercises

Exercise. *Find* $\frac{dy}{dt}$ *when* $y = \sin^2(\pi t - 2)$.

We have, $y = \sin^2(\pi t - 2) = [\sin(\pi t - 2)]^2$, so that:

$$\begin{aligned}
\frac{dy}{dt} &= 2\left(\sin(\pi t - 2)\right)^1 \frac{d}{dt}\left(\sin(\pi t - 2)\right) \text{ since } \frac{dy}{dt}(u^n) = nu^{n-1}\left(\frac{d}{dt}u\right) \\
&= 2\sin(\pi t - 2)\left[\cos(\pi t - 2)\frac{d}{dt}(\pi t - 2)\right] \text{ since } (f \circ g)' = [f'(g)]g' \\
&\qquad [\text{Here } f = \cos \text{ and } g(t) = \pi t - 2] \\
&= 2\sin(\pi t - 2)\cos(\pi t - 2) \cdot \pi \\
&= \pi \sin(2\pi t - 4) \text{ since } \sin(2\theta) = 2(\sin\theta)(\cos\theta)
\end{aligned}$$

Exercise. *Find* y'' *when* $y = \frac{1}{9}\cot(3x - 1)$.

We have,

$$\begin{aligned}
y' &= \frac{1}{9} \cdot \frac{d}{dx}\left[\cot(3x - 1)\right] \text{ since } \frac{d}{dx}(ku) = k\left(\frac{d}{dx}u\right) \text{ for constant } k \\
&= \frac{1}{9}\left[-\csc^2(3x - 1) \cdot \frac{d}{dx}(3x - 1)\right] \text{ since } (f \circ g)' = [f'(g)]g' \\
&\qquad [\text{Here } f = \cot \text{ and } g(x) = 3x - 1] \\
&= -\frac{1}{9}\csc^2(3x - 1) \cdot 3 \\
&= -\frac{1}{3}\csc^2(3x - 1)
\end{aligned}$$

Then,

$$\begin{aligned}
y'' &= \frac{d}{dx}y' = \frac{d}{dx}\left(-\frac{1}{3}\csc^2(3x - 1)\right) \\
&= -\frac{1}{3} \cdot \frac{d}{dx}\left(\csc^2(3x - 1)\right) \\
&= -\frac{1}{3}\left[2\left(\csc(3x - 1)\right)^1 \cdot \frac{d}{dx}\left(\csc(3x - 1)\right)\right] \text{ since } \frac{d}{dx}(u^n) = nu^{n-1}\left(\frac{d}{dx}u\right) \\
&= -\frac{2}{3}\csc(3x - 1)\left[-\csc(3x - 1)\cot(3x - 1) \cdot \frac{d}{dx}(3x - 1)\right] \text{ since } (f \circ g)' = [f'(g)]g' \\
&\qquad [\text{Here } f = \csc \text{ and } g(x) = 3x - 1] \\
&= \frac{2}{3}\csc^2(3x - 1)\cot(3x - 1) \cdot 3 \\
&= 2\csc^2(3x - 1)\cot(3x - 1)
\end{aligned}$$

Exercise. *Use implicit differentiation to find* $\frac{dy}{dx}$*:*

$$
\begin{aligned}
&(3xy+7)^2 = 6y \\
&\implies \frac{d}{dx}\left[(3xy+7)^2\right] = \frac{d}{dx}\left[6y\right] \\
&\implies \frac{d}{dx}\left[(3xy+7)^2\right] = 6y' \text{ since } \frac{d}{dx}(y) = \frac{dy}{dx} = y' \\
&\implies 2(3xy+7)^1 \cdot \frac{d}{dx}(3xy+7) = 6y' \text{ since } \frac{d}{dx}(u^n) = nu^{n-1}\left(\frac{d}{dx}u\right) \\
&\implies 2(3xy+7)\left[\frac{d}{dx}(3xy) + \frac{d}{dx}(7)\right] = 6y' \text{ since } \frac{d}{dx}(u+v) = \left(\frac{d}{dx}u\right) + \left(\frac{d}{dx}v\right) \\
&\implies 2(3xy+7)\left[3\frac{d}{dx}(xy) + 0\right] = 6y' \\
&\qquad \text{since } \frac{d}{dx}(ku) = u\left(\frac{d}{dx}u\right) \text{ and } \frac{d}{dx}(k) = 0, \text{ for constant } k \\
&\implies 2(3xy+7)\left[3\left(x\frac{d}{dx}(y) + y\frac{d}{dx}(x)\right)\right] = 6y' \text{ since } \frac{d}{dx}(uv) = u\left(\frac{d}{dx}v\right) + v\left(\frac{d}{dx}u\right) \\
&\implies 6(3xy+7)(xy'+y) = 6y' \text{ since } \frac{d}{dx}(y) = y' \text{ and } \frac{d}{dx}(x) = 1 \\
&\implies (3xy+7)(xy'+y) = y'
\end{aligned}
$$

And now we need use only algebra to solve for y':

$$
\begin{aligned}
&\implies 3x^2yy' + 3xy^2 + 7xy' + 7y = y' \\
&\implies 3x^2yy' + 7xy' - y' = -3xy^2 - 7y \\
&\implies (3x^2y + 7x - 1)y' = -3xy^2 - 7y \\
&\implies y' = \frac{-3xy^2 - 7y}{3x^2y + 7x - 1}
\end{aligned}
$$

Exercise. *Verify that the point* $(\frac{\pi}{4}, \frac{\pi}{2})$ *is on the curve* $x \sin 2y = y \cos 2x$ *and find the lines that are tangent and normal to this curve at this point.*

When $(x, y) = (\frac{\pi}{4}, \frac{\pi}{2})$, we have $x \sin 2y = \frac{\pi}{4} \sin 2\frac{\pi}{2} = 0$ and $y \cos 2x = \frac{\pi}{2} \cos 2\frac{\pi}{4} = 0$, and therefore $x \sin 2y = y \cos 2x$ is satisfied. Therefore the point $(\frac{\pi}{4}, \frac{\pi}{2})$ is indeed on this curve.

To find an equation for the tangent line at this point, we need its slope, which is simply the value of y' when $(x, y) = (\frac{\pi}{4}, \frac{\pi}{2})$. Implicitly differentiating the equation for the curve, we get,

$$\begin{aligned}
&\frac{d}{dx}\left[x \sin 2y\right] = \frac{d}{dx}\left[y \cos 2x\right] \\
&\Longrightarrow x \cdot \frac{d}{dx}(\sin 2y) + (\sin 2y) \cdot \frac{d}{dx}(x) = y \cdot \frac{d}{dx}(\cos 2x) + (\cos 2x) \cdot \frac{d}{dx}(y) \\
&\qquad \text{since } \frac{d}{dx}(uv) = u\left(\frac{d}{dx}v\right) + v\left(\frac{d}{dx}u\right) \\
&\Longrightarrow x \cdot \frac{d}{dx}(\sin 2y) + (\sin 2y) = y \cdot \frac{d}{dx}(\cos 2x) + (\cos 2x)y' \\
&\qquad \text{since } \frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = y' \\
&\Longrightarrow x\left[(\cos 2y) \cdot \frac{d}{dx}(2y)\right] + (\sin 2y) = y\left[(-\sin 2x) \cdot \frac{d}{dx}(2x)\right] + (\cos 2x)y' \\
&\qquad \text{since } (f \circ g)' = [f'(g)]g' \\
&\Longrightarrow x(\cos 2y)\left[2 \cdot \frac{d}{dx}(y)\right] + (\sin 2y) = -y(\sin 2x)\left[2 \cdot \frac{d}{dx}(x)\right] + (\cos 2x)y' \\
&\qquad \text{since } \frac{d}{dx}(ku) = k\left(\frac{d}{dx}u\right) \text{ for constant } k \\
&\Longrightarrow 2x(\cos 2y)y' + (\sin 2y) = -2y(\sin 2x) + (\cos 2x)y' \\
&\qquad \text{since } \frac{d}{dx}(y) = y' \text{ and } \frac{d}{dx}(x) = 1
\end{aligned}$$

Therefore at $(x, y) = (\frac{\pi}{4}, \frac{\pi}{2})$ we have,

$$\begin{aligned}
&2\left(\frac{\pi}{4}\right)(\cos 2\frac{\pi}{2})y' + (\sin 2\frac{\pi}{2}) = -2\left(\frac{\pi}{2}\right)\left(\sin 2\frac{\pi}{4}\right) + \left(\cos 2\frac{\pi}{4}\right)y' \\
&\Longrightarrow \frac{\pi}{2}(-1)y' + 0 = -\pi + 0 \\
&\Longrightarrow -\frac{\pi}{2}y' = -\pi \\
&\Longrightarrow y' = (-\pi)\left(-\frac{2}{\pi}\right) = 2
\end{aligned}$$

so that the tangent line at this point is $y - \frac{\pi}{2} = 2\left(x - \frac{\pi}{4}\right)$. And the normal is simply perpendicular to the tangent, therefore its equation is $y - \frac{\pi}{2} = -\frac{1}{2}\left(x - \frac{\pi}{4}\right)$.