Modern Physics: Chapter 4 Homework

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1. *(book #5) What is the shortest wavelength present in the Brackett series of spectral lines?*

The shortest wavelength present in the Brackett series corresponds to the limit of

$$\frac{1}{\lambda} = R\left(\frac{1}{4^2} - \frac{1}{n^2}\right)$$

as n approaches infinity where $R = 1.097 \times 10^7 \mathrm{m}^{-1}$.

Then,

$$\begin{aligned} \frac{1}{\lambda} &= \frac{1}{16}R \\ \Longrightarrow \lambda &= \frac{16}{R} = \frac{16}{1.097 \times 10^7}\mathrm{m} \\ &= 1.459 \times 10^{-6}\mathrm{m} = \boxed{1.459\mu\mathrm{m}} \end{aligned}$$

2. *(book #9) The* **fine structure constant** *is defined as $\alpha = e^2/2\epsilon_0 hc$. This quantity got its name because it first appeared in a theory by the German physicist Arnold Sommerfeld that tried to explain the fine structure in spectral lines (multiple lines close together instead of single lines) by assuming that elliptical as well as circular orbits are possible in the Bohr model. Sommerfeld's approach was on the wrong track, but α has nevertheless turned out to be a useful quantity in atomic physics.*

(a) *Show that $\alpha = \nu_1/c$, where ν_1 is the velocity of the electron in the ground state of the Bohr atom.*

The velocity ν of an electron orbiting a nucleus at a radius r is given by

$$\nu = \frac{e}{\sqrt{4\pi\epsilon_0 m_e r}}$$

For an electron in the ground state, one de Broglie wavelength of the electron must be equal to the circumference of its orbit, so that letting ν_1 represent the velocity of the electron in the ground state we have

$$\begin{aligned} \lambda &= \frac{h}{\gamma m_e \nu_1} = 2\pi r \\ \Longrightarrow r &= \frac{h}{2\pi\gamma m_e \nu_1} \end{aligned}$$

Substituting this expression for r into our formula for the velocity of the electron gives us the velocity of the electron in the ground state, hence

$$\nu_1 = \frac{e}{\sqrt{4\pi\epsilon_0 m_e \frac{h}{2\pi\gamma m_e \nu_1}}}$$
$$= \frac{e\sqrt{\gamma\nu_1}}{\sqrt{2\epsilon_0 h}}$$
$$\Longrightarrow \nu_1^2 = \frac{e^2\gamma\nu_1}{2\epsilon_0 h}$$
$$\Longrightarrow \nu_1 = \frac{e^2\gamma}{2\epsilon_0 h}$$

Therefore, ignoring relativistic effects (hence $\gamma \approx 1$),

$$\alpha = \frac{e^2}{2\epsilon_0 hc} = \frac{\nu_1}{c}$$

Q.E.D.

(b) *Show that the value of α is very close to $1/137$ and is a pure number with no dimensions. Because the magnetic behavior of a moving charge depends on its velocity, the small value of α is representative of the relative magnitudes of the magnetic and electric aspects of electron behavior in an atom.*

$$\alpha = \frac{e^2}{2\epsilon_0 hc}$$
$$= \frac{(1.602 \times 10^{-19}\text{C})^2}{2(8.854 \times 10^{-12}\text{C}^2/\text{N}\cdot\text{m}^2)(6.626 \times 10^{-34}\text{J}\cdot\text{s})(2.998 \times 10^{8}\text{m/s})}$$
$$= 7.296 \times 10^{-3}\text{J/N}\cdot\text{m}$$
$$= \frac{1}{137.1}$$
$$\approx \frac{1}{137}$$

Q.E.D.

(c) *Show that $\alpha a_0 = \lambda_C/2\pi$, where a_0 is the radius of the ground-state Bohr orbit and λ_C is the Compton wavelength of the electron.*

From (a) we have that

$$\alpha = \frac{\nu_1}{c} \quad \text{and} \quad a_0 = \frac{h}{2\pi m_e \nu_1}$$

Therefore,

$$\alpha a_0 = \frac{\nu_1}{c} \cdot \frac{h}{2\pi m_e \nu_1} = \frac{h}{2\pi m_e c}$$
$$= \frac{\lambda_C}{2\pi}$$

since $\lambda_C = \frac{h}{m_e c}$. **Q.E.D.**

3. *(book #19) Find the wavelength of the spectral line that corresponds to a transition in hydrogen from the $n = 10$ state to the ground state. In what part of the spectrum is this?*

We have

$$\begin{aligned}\frac{1}{\lambda} &= R\left(\frac{1}{1^2} - \frac{1}{10^2}\right) = \frac{99}{100}R \\ \implies \lambda &= \frac{100}{99R} = \frac{100}{99(1.097 \times 10^7 \mathrm{m}^{-1})} \\ &= 9.21 \times 10^{-8}\mathrm{m} = \boxed{92.1\mathrm{nm}}\end{aligned}$$

This lies in the $\boxed{\text{Ultraviolet}}$ part of the spectrum.

4. *(book #21) A beam of electrons bombards a sample of hydrogen. Through what potential difference must the electrons have been accelerated if the first line of the Balmer series is to be emitted?*

The first line of the Balmer series corresponds to an energy level change from $n = 3$ to $n = 2$. Therefore the bombardment must excite the electrons in the hydrogen atoms from the ground state $n = 1$ to the $n = 3$ state at a minimum. This corresponds to an energy of

$$\frac{E_1}{3^2} - \frac{E_1}{1^2} = \frac{-13.6\mathrm{eV}}{9} + 13.6\mathrm{eV} = \boxed{12.1\mathrm{eV}}$$

5. *(book #22) How much energy is required to remove an electron in the $n = 2$ state from a hydrogen atom.*

The minimum amount of energy required is the energy required to increase the electron's energy to zero:

$$0 - \frac{E_1}{2^2} = -\frac{-13.6\mathrm{eV}}{4} = \boxed{3.4\mathrm{eV}}$$

6. *(book #25) An excited hydrogen atom emits a photon of wavelength λ in returning to the ground state.*

(a) *Derive a formula that gives the quantum number of the initial excited state in terms of λ and R.*

We have,

$$\frac{1}{\lambda} = R\left(\frac{1}{1^2} - \frac{1}{n_i^2}\right) = R - \frac{R}{n_i^2}$$
$$\Longrightarrow \frac{1}{\lambda} - R = -\frac{R}{n_i^2}$$
$$\Longrightarrow \frac{1}{n_i^2} = 1 - \frac{1}{\lambda R} = \frac{\lambda R - 1}{\lambda R}$$
$$\Longrightarrow n_i = \boxed{\sqrt{\frac{\lambda R}{\lambda R - 1}}}$$

(b) *Use this formula to find n_i for a 102.55nm photon.*

$$\lambda R = (102.55 \times 10^{-9}\text{m})(1.097 \times 10^{7}\text{m}^{-1}) = 1.125 = 9/8$$

Therefore

$$n_i = \sqrt{\frac{9/8}{1/8}} = \sqrt{9} = \boxed{3}$$

7. *(book #31) A μ^- muon is in the $n = 2$ state of a muonic atom whose nucleus is a proton. Find the wavelength of the photon emitted when the muonic atom drops to its ground state. In what part of the spectrum is this wavelength?*

We have

$$\frac{1}{\lambda} = -\frac{E_1'}{ch}\left(\frac{1}{1^2} - \frac{1}{2^2}\right)$$
$$= -\frac{3E_1'}{4ch}$$
$$\Longrightarrow \lambda = -\frac{4ch}{3E_1'}$$

where E_1' for the muon is given relative to the reduced mass m' of the muon by

$$E_1' = -\frac{m'e^4}{8\epsilon_0^2 h^2}$$

We have the mass of the muon as $m = 207m_e$ and the mass of the nucleus as $M = 1836m_e$ so that the reduced mass m' of the muon is

$$m' = \frac{mM}{m+M} = \frac{(207m_e)(1836m_e)}{207m_e + 1836m_e} = 186m_e$$

so that

$$
\begin{aligned}
E_1' &= -\frac{186(9.109\times10^{-31}\mathrm{kg})(1.602\times10^{-19}\mathrm{C})^4}{8(8.854\times10^{-12}\mathrm{C}^2/\mathrm{N}\cdot\mathrm{m}^2)^2(6.626\times10^{-34}\mathrm{J}\cdot\mathrm{s})^2} \\
&= -\frac{186(9.109)(1.602)^4}{8(8.854)^2(6.626)^2}\cdot\frac{(10^{-31}\mathrm{kg})(10^{-76}\mathrm{C}^4)}{(10^{-24}\mathrm{C}^4/\mathrm{N}^2\cdot\mathrm{m}^4)(10^{-68}\mathrm{J}^2/\mathrm{s}^2)} \\
&= -0.4053\times10^{-15}\frac{\mathrm{kg}\cdot\mathrm{N}^2\cdot\mathrm{m}^4}{\mathrm{J}^2\cdot\mathrm{s}^2} \\
&= -4.053\times10^{-16}\frac{\mathrm{kg}^3\cdot\mathrm{m}^6/\mathrm{s}^4}{\mathrm{kg}^2\cdot\mathrm{m}^4/\mathrm{s}^2} \\
&= -4.053\times10^{-16}\mathrm{J}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
\lambda &= -\frac{4ch}{3E_1'} \\
&= -\frac{4(2.998\times10^{8}\mathrm{m/s})(6.626\times10^{-34}\mathrm{J}\cdot\mathrm{s})}{3(-4.053\times10^{-16}\mathrm{J})} \\
&= 6.53\times10^{-10}\mathrm{m}
\end{aligned}
$$

Hence the wavelength of the photon emitted when the muonic atom drops to its ground state will be $\boxed{0.653\mathrm{nm}}$ which corresponds to an $\boxed{\text{X-ray}}$.