Techniques of Constructive Analysis

Let $\bar{x}$ and $\bar{y}$ be real numbers.

Definition. *$\bar{x} = \bar{y}$ if for all $(q, q') \in \bar{x}$ and all $(r, r') \in \bar{y}$, the intervals $[q, q']$ and $[r, r']$ in $\mathbf{Q}$ have a rational point in common.*

Definition. *$\bar{x} \neq \bar{y}$ if there exist $(q, q') \in \bar{x}$ and $(r, r') \in \bar{y}$ such that the intervals $[q, q']$ and $[r, r']$ in $\mathbf{Q}$ are disjoint.*

Definition. *$\bar{x} < \bar{y}$ if there exist $(q, q') \in \bar{x}$ and $(r, r') \in \bar{y}$ such that $q' < r$.*

Definition. $xy = \{\min(qr, qr', q'r, q'r'), \max(qr, qr', q'r, q'r') : (q, q') \in \bar{x}, (r, r') \in \bar{y}\}$

Theorem. *If $\bar{x} \neq \bar{0}$ then there exists a real number $1/\bar{x}$ such that $\bar{x}(1/\bar{x}) = \bar{1}$.*

Suppose $\bar{x} > \bar{0}$. Then define $1/\bar{x} = \{(1/q', 1/q) : (q, q') \in \bar{x}$ such that $q > 0\}$.

Since $\bar{x} > \bar{0}$, there exists $(q_0, q_0') \in \bar{x}$ with $q_0 > 0$. Therefore $1/\bar{x}$ is inhabitated. And $0 < q \leq q' \implies \frac{1}{q'} \leq \frac{1}{q}$.

Now suppose $(1/q', 1/q), (1/r', 1/r) \in 1/\bar{x}$. Then since $[q, q'], [r, r']$ overlap in $\mathbf{Q}$, we have $0 < q \leq r' \implies 1/r' \leq 1/q$ and $0 < r \leq q' \implies 1/q' \leq 1/r$. Therefore any two pairs taken in $1/\bar{x}$ overlap as intervals in $\mathbf{Q}$.

Now choose rational $\epsilon > 0$. Since $q_0 > 0$, there exists a positive integer n such $q_0 > \frac{1}{n}$. Since all pairs in $\bar{x}$ overlap with $[q_0, q_0']$ as closed intervals in $\mathbf{Q}$, there must exist $(q, q') \in \bar{x}$ with $q > \frac{1}{n}$ and $q' - q < \epsilon$ for every positive rational ϵ. (Every $(q, q') \in \bar{x}$ with $q' - q < q_0 - \frac{1}{n}$ must have $q > \frac{1}{n}$ since $q_0 \leq q'$.) Find $(q, q') \in \bar{x}$ such that $q > \frac{1}{n}$ and $q' - q < \epsilon/n^2$. Then $(\frac{1}{q'}, \frac{1}{q}) \in 1/\bar{x}$, and

$$\begin{aligned}
\frac{1}{q} - \frac{1}{q'} &= \frac{q' - q}{qq'} \\
&\leq \frac{q' - q}{1/n^2} \text{ since } \frac{1}{n} \leq q \leq q' \\
&= n^2(q' - q) \\
&\leq n^2 \left(\frac{\epsilon}{n^2}\right) \\
&= \epsilon
\end{aligned}$$

Therefore there exists $(s, s') \in \frac{1}{\bar{x}}$ such that $s' - s < \epsilon$ for every positive rational ϵ.

To show $\bar{x}(1/\bar{x}) = \bar{1}$, we will show that $s \leq 1 \leq s'$ for every $(s, s') \in \bar{x}(1/\bar{x})$. We have,

$$\bar{x}\left(\frac{1}{\bar{x}}\right) = \left\{\left(\min\left(\frac{q}{r}, \frac{q'}{r}, \frac{q}{r'}, \frac{q'}{r'}\right), \max\left(\frac{q}{r}, \frac{q'}{r}, \frac{q}{r'}, \frac{q'}{r'}\right)\right) : (q, q'), (r, r') \in \bar{x},\ r > 0\right\}$$

But $q \leq r'$ and $r \leq q'$ so that $\frac{q}{r'} \leq 1$ and $\frac{q'}{r} \geq 1$ and hence $s \leq 1 \leq s'$ for every $(s, s') \in \bar{x}(1/\bar{x})$. Therefore $\bar{x}(1/\bar{x}) = \bar{1}$.

If $\bar{x} < \bar{0}$ then define $1/\bar{x} = -(1/(-\bar{x}))$. Then $\bar{x}(1/\bar{x}) = (-\bar{x})(1/(-\bar{x})) = 1$ by the above. **Q.E.D.**

Proposition. The statement "All functions from a linear subspace of $\mathbf{R}^\infty$ to $\mathbf{R}$ are strongly extensional" implies Markov's Principle.

Suppose the statement "All functions from a linear subspace of $\mathbf{R}^\infty$ to $\mathbf{R}$ are strongly extensional" were true. Let $a = (a_n)$ be a binary sequence such that it is impossible that for all n, $a_n = 0$. We need to show $a \neq 0$ and hence Markov's Principle obtains.

Let $f : \mathbf{R}a \to \mathbf{R}$ be defined by $f(xa) = x$. To see that f is a well-defined function, suppose $xa = ya$ for some $x, y \in \mathbf{R}$. Then $|x - y| > 0$ contradicts $xa = ya$ since it is impossible that $a = 0$. Therefore it is impossible that $|x - y| > 0$ and hence $|x - y| = 0$ so that $x = y$. Therefore f is a well-defined function. Now since $f(1a) = 1$ and $f(0a) = 0$ and $1 \neq 0$, it follows from the strong extensionality of f (by hypothesis) that $1a \neq 0a$ and hence $a \neq 0$. From this Markov's Principle follows. **Q.E.D.**

Lemma. For $\mathbf{x}, \mathbf{y} \in \mathbf{R}$, $\mathbf{xy} \neq 0 \Longrightarrow (\mathbf{x} \neq 0 \wedge \mathbf{y} \neq 0)$.

Suppose $\mathbf{x}, \mathbf{y} \in \mathbf{R}$ such that $\mathbf{xy} \neq 0$. Then there exists $(s, s') \in \mathbf{xy}$ such that $0 \notin [s, s']$. And then there exists $(q, q') \in \mathbf{x}$ and $(r, r') \in \mathbf{y}$ such that $s = \min\{qr, qr', q'r, q'r'\}$ and $s' = \max\{qr, qr', q'r, q'r'\}$. Now since q,q',r, and r' are all rational, we can argue as follows:

Suppose $q \leq 0$ and $q' \geq 0$. Then at least one of the following holds:

$$r \leq 0 \wedge r' \geq 0 \Longrightarrow qr' \leq 0 \leq q'r' \Longrightarrow 0 \in [s, s']$$
$$r \leq 0 \wedge r' \leq 0 \Longrightarrow q'r \leq 0 \leq qr \Longrightarrow 0 \in [s, s']$$
$$r \geq 0 \wedge r' \geq 0 \Longrightarrow qr \leq 0 \leq q'r \Longrightarrow 0 \in [s, s']$$

all of which contradict. Therefore either $q > 0$ or $q' < 0$ and hence $\mathbf{x} \neq 0$.

Likewise $\mathbf{y} \neq 0$ by a symmetric argument. Therefore $\mathbf{xy} \neq 0 \Longrightarrow (\mathbf{x} \neq 0 \wedge \mathbf{y} \neq 0)$. **Q.E.D.**

Proposition. The statement "every linear subspace of $\mathbf{R}$ is either $\{0\}$ or $\mathbf{R}$" implies LPO.

Suppose the statement "every linear subspace of $\mathbf{R}$ is either $\{0\}$ or $\mathbf{R}$" were true, and let (a_n) be any binary sequence. Let $\mathbf{x} \subset \mathbf{Q} \times \mathbf{Q}$ be defined by

$$\mathbf{x} = \left\{ \left(0, \frac{1}{n}\right) : a_n = 0 \right\} \cup \left\{ \left(\frac{1}{n}, \frac{1}{n}\right) : a_n = 1 \wedge (a_k = 0 \text{ for all } k < n) \right\}$$

Since (a_n) is a binary sequence, for any finite n we can determine either $a_k = 0$ for all $k \leq n$ or else there exists $k \leq n$ such that $a_k = 1$. Therefore $\mathbf{x}$ is inhabited. And for any rational $\epsilon > 0$, n exists such that $\frac{1}{n} < \epsilon$ so that we have either $(0, \frac{1}{n}) \in \mathbf{x}$ or $(\frac{1}{k}, \frac{1}{k}) \in \mathbf{x}$ for some $k \leq n$. Furthermore suppose (q, q') and (r, r') distinct in $\mathbf{x}$ are chosen. Then $q' = \frac{1}{n_q}$ and $r' = \frac{1}{n_r}$ for some distinct n_q, n_r. Without loss of generality we may assume $n_r > n_q$. Then $q = 0$ since $q > 0 \Longrightarrow q = q' \Longrightarrow n_q > n_r$ contradicts. Therefore $q \leq r' < q'$ so that $[q, q']$ and $[r, r']$ intersect in $\mathbf{Q}$. Therefore $\mathbf{x}$ is a real number.

Now consider the linear subspace $\mathbf{Rx}$. Either $\mathbf{Rx} = \{0\}$ or $\mathbf{Rx} = \mathbf{R}$ by hypothesis. If $\mathbf{Rx} = \{0\}$ then $1\mathbf{x} = 0 \Longrightarrow \mathbf{x} = 0 \Longrightarrow 0 \in [q, q']$ for all $(q, q') \in \mathbf{x}$ and hence $a_n = 0$ for all n. Otherwise, $\mathbf{Rx} = \mathbf{R}$ and then $\mathbf{yx} = 1$ for some $\mathbf{y} \in \mathbf{R}$. But then $\mathbf{x} \neq 0$ by Lemma and hence n exists such that $a_n = 1$. Therefore LPO follows. **Q.E.D.**