Elementary Algebra Exercises

Exercise. *A total of* $20,000 *is going to be split between Adam and Krissy with Adam receiving* $3000 *less than Krissy. How much will each get?*

Let k be the amount Krissy will get. I chose to label Krissy's amount rather than Adam's since Adam's amount has been expressed relative to Krissy's (Adam will receive $k - 3000$).

Now we know that the sum of what Krissy and Adam get is 20,000. Therefore the following are equivalent:

$$\begin{aligned}
20000 &= k + (k - 3000) \\
20000 &= k + k + (-3000) \quad \boxed{\text{since Addition is Associative}} \\
20000 &= 2k + (-3000) \\
20000 + 3000 &= 2k + (-3000) + 3000 \quad \boxed{\text{by adding 3000 to both sides}} \\
23000 &= 2k + 0 \\
23000 &= 2k \quad \boxed{\text{since 0 is the Additive Identity}} \\
\frac{1}{2}23000 &= \frac{1}{2}2k \quad \boxed{\text{by multiplying both sides by } \tfrac{1}{2} \text{ (or dividing by 2 if you prefer).}} \\
11500 &= 1 \cdot k \\
11500 &= k \quad \boxed{\text{since 1 is the Multiplicative Identity}}
\end{aligned}$$

Therefore Krissy must receive \$11,500, and since Adam receives \$3000 less than this, Adam must receive \$8500.

Exercise. *Patrick is a loan officer at a bank. He has* $2,000,000 *to lend out and has two loan programs. His home equity loan is currently priced at* 6% *per annum, while his unsecured personal loan is priced at* 14%. *The bank president wants Patrick to earn a rate of return of* 12% *on the* $2,000,000 *available. How much should Patrick lend out at* 6%*?*

Let h (for "home") be the amount Patrick should lend out in home equity loans (at an interest rate of 0.06). Now we have to assume that Patrick will loan out the rest of the 2,000,000 available in unsecured personal loans at an interest rate of 0.14. (We often have to make such assumptions when solving problems). Therefore he will loan out $2,000,000 - h$ in unsecured personal loans. The interest that will accrue for the home equity loans will therefore be $0.06h$ and the interest that will accrue for the unsecured personal loans will be $0.14(2,000,000 - h)$.

Now, what the bank president wants is for Patrick to do the equivalent of lending out all the money at an interest rate of 0.12. Therefore the total accrued interest should be equivalent to $2,000,000 \cdot 0.12$.

Then the following are equivalent:

$$\begin{aligned}
0.06h + 0.14(2{,}000{,}000 - h) &= 2{,}000{,}000 \cdot 0.12 \\
0.06h + 0.14(2{,}000{,}000 + (-h)) &= 240{,}000 \\
0.06h + (0.14)(2{,}000{,}000) + (0.14)(-h) &= 240{,}000 \quad \boxed{\text{by the Distributive Law}} \\
0.06h + 280{,}000 + (-0.14)h &= 240{,}000 \quad \boxed{\text{by simplifying}} \\
0.06h + (-0.14)h + 280{,}000 &= 240{,}000 \quad \boxed{\text{since Addition is Commutative}} \\
(0.06 + (-0.14))h + 280{,}000 &= 240{,}000 \quad \boxed{\text{by the Distributive Law}} \\
-0.08h + 280{,}000 &= 240{,}000 \quad \boxed{\text{by simplifying}} \\
-0.08h + 280{,}000 + (-280{,}000) &= 240{,}000 + (-280{,}000) \quad \boxed{\text{adding to both sides}} \\
-0.08h + 0 &= -40{,}000 \quad \boxed{\text{by simplifying}} \\
-0.08h &= -40{,}000 \quad \boxed{\text{since 0 is the Additive Identity}} \\
\left(\frac{1}{-0.08}\right)(-0.08)h &= \left(\frac{1}{-0.08}\right)(-40{,}000) \quad \boxed{\text{multiplying both sides}} \\
1 \cdot h &= 500{,}000 \quad \boxed{\text{by simplifying}} \\
h &= 500{,}000
\end{aligned}$$

Therefore if Patrick lends out \$500,000 in home equity loans (at 6% interest) and lends out the rest at 14% then Patrick will earn a rate of return of 12% on the \$2,000,000.

Exercise. *The purity of gold is measured in karats, with pure gold being 24 karats. Other purities of gold are expressed as proportional parts of pure gold. For example, 18-karat gold is $\frac{18}{24}$, or 75%, pure gold; 12-karat gold is $\frac{12}{24}$, or 50%, pure gold; and so on. How much pure gold should be mixed with 12-karat gold to obtain 72 grams of 18-karat gold?*

Once our gold is mixed, we need there to be $(\frac{18}{24})72$ grams of pure gold in the result in order for it to be 18-karat gold. That would be $(\frac{3}{4})72$ or 54 grams of gold in the result. So if we use g (for "gold") grams of pure gold, the remaining $72 - g$ grams will be 12-karat gold, and the total pure gold content will therefore be $g + (\frac{12}{24})(72 - g)$ grams. Therefore we require the following equivalent equations to be satisfied:

$$
\begin{aligned}
g + \left(\frac{12}{24}\right)(72 - g) &= 54 \\
g + \left(\frac{1}{2}\right)(72 + (-g)) &= 54 \quad \boxed{\text{by simplifying}} \\
g + \left(\frac{1}{2}\right)(72) + \left(\frac{1}{2}\right)(-g) &= 54 \quad \boxed{\text{by the Distributive Law}} \\
g + 36 + \left(-\frac{1}{2}\right) g &= 54 \quad \boxed{\text{by simplifying}} \\
g + \left(-\frac{1}{2}\right) g + 36 &= 54 \quad \boxed{\text{since Addition is Commutative}} \\
1 \cdot g + \left(-\frac{1}{2}\right) g + 36 &= 54 \quad \boxed{\text{since 1 is the Multiplicative Identity}} \\
\left(1 + \left(-\frac{1}{2}\right)\right) g + 36 &= 54 \quad \boxed{\text{by the Distributive Law}} \\
\frac{1}{2} g + 36 &= 54 \quad \boxed{\text{by simplifying}} \\
\frac{1}{2} g + 36 + (-36) &= 54 + (-36) \quad \boxed{\text{by adding -36 to both sides}} \\
\frac{1}{2} g + 0 &= 18 \quad \boxed{\text{by simplifying}} \\
\frac{1}{2} g &= 18 \quad \boxed{\text{since 0 is the Additive Identity}} \\
2\left(\frac{1}{2} g\right) &= 2(18) \quad \boxed{\text{by multiplying both sides by 2}} \\
1 \cdot g &= 36 \quad \boxed{\text{by simplifying}} \\
g &= 36 \quad \boxed{\text{since 1 is the Multiplicative Identity}}
\end{aligned}
$$

Therefore by using 36 grams of pure gold in the mixture, the result will be 18-karat gold.

Exercise. *Find all solutions for x of the equation $-3x - 4 = 6$.*

We have,

$$
\begin{aligned}
& -3x - 4 = 6 \\
\iff & -3x - 4 + 4 = 6 + 4 \\
\iff & -3x = 10 \\
\iff & \left(-\frac{1}{3}\right)(-3x) = \left(-\frac{1}{3}\right)(10) \\
\iff & x = -\frac{10}{3}
\end{aligned}
$$

Therefore the solution set is $\{-\frac{10}{3}\}$.

Exercise. *Determine whether* $y = \frac{2}{5}$ *is a solution of the equation* $5y - 2 = 3$.

$\frac{2}{5}$ is not a solution since $5(\frac{2}{5}) - 2 = 3$ is not a true statement. (The left hand side would actually equal 0).

Exercise. *Identify each of the following as an identity, a contradiction, or a conditional statement.*

(a) $3 - 5y = 2$

This is a conditional statement since

$$\begin{aligned} & 3 - 5y = 2 \\ \Longleftrightarrow & (-3) + 3 - 5y = 2 + (-3) \\ \Longleftrightarrow & -5y = -1 \\ \Longleftrightarrow & \left(-\frac{1}{5}\right)(-5y) = \left(-\frac{1}{5}\right)(-1) \\ \Longleftrightarrow & y = \frac{1}{5} \text{ has solution set } \{\tfrac{1}{5}\} \\ & \text{(which is neither the empty set nor the set of all real numbers).} \end{aligned}$$

(b) $2z - 1 = 2z$

This is a contradiction since

$$\begin{aligned} & 2z - 1 = 2z \\ \Longleftrightarrow & (-2z) + 2z - 1 = 2z + (-2z) \\ \Longleftrightarrow & -1 = 0 \text{ has solution set } \{\}. \end{aligned}$$

(c) $3x - 1 = x - 1 + 2x$

This is an identity since

$$\begin{aligned} & 3x - 1 = x - 1 + 2x \\ \Longleftrightarrow & 3x - 1 = x + 2x - 1 \\ \Longleftrightarrow & 3x - 1 = 3x - 1 \text{ has solution set of all real numbers.} \end{aligned}$$

Exercise. *Evaluate.*

$$\frac{4 - (8 + 1) - 3 \cdot 5^3}{-3^3 + (-2)^2 + 5} + \left(\frac{2}{3}\right)^2$$

We have,

$$
\begin{aligned}
&\frac{4-(8+1)-3\cdot 5^3}{-3^3+(-2)^2+5}+\left(\frac{2}{3}\right)^2\\
=&\frac{4-9-3\cdot 125}{-27+4+5}+\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\\
=&\frac{4-9-375}{-18}+\frac{4}{9}\\
=&\frac{-380}{-18}+\frac{4}{9}\\
=&\frac{380}{18}+\frac{8}{18}\\
=&\frac{380+8}{18}\\
=&\frac{388}{18}\\
=&\frac{194}{9}
\end{aligned}
$$

Exercise. *Find all solutions for t in the equation $-\frac{1}{3}t-2\geq 4$ and express the solution set in interval notation.*

We have,

$$
\begin{aligned}
&-\frac{1}{3}t-2\geq 4\\
\iff&-\frac{1}{3}t-2+2\geq 4+2\\
\iff&-\frac{1}{3}t\geq 6\\
\iff&(-3)\left(-\frac{1}{3}t\right)\leq(-3)6 \text{ [reversing the direction of the inequality since } -3 \text{ is negative]}\\
\iff&t\leq -18
\end{aligned}
$$

Therefore the solution set is $(-\infty,-18]$.

Exercise. *Suppose we are given the equation $3y+x=-10$.*

(a) **Is $(-1,-3)$ a point on its graph?**

Yes, $(-1,-3)$ is a point on the graph of $3y+x=-10$ since when we substitute -1 for x and -3 for y, we get $3(-3)+(-1)=-10$ which is true.

(b) **Is $(-9,\frac{1}{3})$ a point on its graph?**

No, $(-9,\frac{1}{3})$ is not a point of the graph of $3y+x=-10$ since when we substitute -9 for x and $\frac{1}{3}$ for y, we get $3(\frac{1}{3})+(-9)=-10$ which is false.

Exercise. *Put $3x-\frac{4}{3}y=4$ in Slope-Intercept form.*

We have,

$$
\begin{aligned}
&3x - \frac{4}{3}y = 4\\
\iff &3x - \frac{4}{3}y + \frac{4}{3}y = 4 + \frac{4}{3}y\\
\iff &3x = 4 + \frac{4}{3}y\\
\iff &3x + (-4) = (-4) + 4 + \frac{4}{3}y\\
\iff &3x + (-4) = \frac{4}{3}y\\
\iff &\frac{3}{4}(3x + (-4)) = \left(\frac{3}{4}\right)\left(\frac{4}{3}\right)y\\
\iff &\left(\frac{3}{4}\right)(3x) + \left(\frac{3}{4}\right)(-4) = y\\
\iff &\frac{9}{4}x + (-3) = y
\end{aligned}
$$

Therefore we have the Slope-Intercept form, $y = \frac{9}{4}x + (-3)$.

Exercise. *Find an equation for the straight line passing through the point* $(\frac{3}{4}, -2)$ *having slope* -1*.*

We are given a point and a slope, so we can use the Point-Slope form of a line to obtain an equation: $y - (-2) = -1(x - \frac{3}{4})$.

Equivalent equations would therefore be:

$$
\begin{aligned}
&y + 2 = x + \frac{3}{4}\\
\iff &y = x + \frac{3}{4} - 2\\
\iff &y = x - \frac{5}{4}
\end{aligned}
$$

(Any of which, as well as other equivalent equations, would be correct answers to this question).

Exercise. *Find an equation for the straight line passing through the points* $(1, 7)$ *and* $(-2, 0)$.

We need a slope to use the Point-Slope form of a line, but we can get the slope easily from these two points using the definition of slope.

We have $m = \frac{0-7}{-2-1} = \frac{-7}{-3} = \frac{7}{3}$.

Then we can use the Point-Slope form with either of the two points to get an equation for the line.

If we use $(1, 7)$ then we get $y - 7 = \frac{7}{3}(x - 1)$.

If we use $(-2, 0)$ then we get $y - 0 = \frac{7}{3}(x - (-2))$ or, equivalently, $y = \frac{7}{3}(x + 2)$.

Either of these, as well as other equivalent equations, would be correct answers to this question.

Exercise. *Find an equation for the line passing through the point* $(5,4)$ *and perpendicular to the line given by* $3x - y = \frac{2}{3}$.

We need to know the slope of the given line before we can determine the slope of a line perpendicular to it. To find the slope, let's first put $3x - y = \frac{2}{3}$ in Slope-Intercept form. We have,

$$\begin{aligned}
&3x - y = \frac{2}{3} \\
\Longleftrightarrow &3x - y + y = \frac{2}{3} + y \\
\Longleftrightarrow &3x = \frac{2}{3} + y \\
\Longleftrightarrow &3x + \left(-\frac{2}{3}\right) = \left(-\frac{2}{3}\right) + \frac{2}{3} + y \\
\Longleftrightarrow &3x - \frac{2}{3} = y \\
\Longleftrightarrow &y = 3x - \frac{2}{3} \text{ simply by interchanging the left and right sides.}
\end{aligned}$$

Therefore the slope of the given line is 3 (the coefficient of x in Slope-Intercept form) and hence every line perpendicular to it will have slope $-\frac{1}{3}$ (the negative reciprocal of 3).

Therefore, using Point-Slope form we obtain the equation of the line passing through the point $(5,4)$ and perpendicular to the given line: $y - 4 = -\frac{1}{3}(x-5)$

This equation or any equation equivalent to it would be a correct answer to this question.

Exercise. *The lateral surface area for a frustum of a right circular cone is given by* $A = \pi s(R+r)$ *where* s *is the slant height of the frustum,* R *is the radius of the base, and* r *is the radius of the top.*

(a) **Solve the equation for** r.

We have,

$$\begin{aligned}
&A = \pi s(R+r) \\
\Longleftrightarrow &A\left(\frac{1}{\pi s}\right) = \left(\frac{1}{\pi s}\right)\pi s(R+r) \\
\Longleftrightarrow &\frac{A}{\pi s} = R + r \\
\Longleftrightarrow &\frac{A}{\pi s} + (-R) = (-R) + R + r \\
\Longleftrightarrow &\frac{A}{\pi s} - R = r
\end{aligned}$$

Therefore $r = \frac{A}{\pi s} - R$. Another way to write this would be $r = \frac{A - \pi s R}{\pi s}$.

(b) **If the frustum of a right circular cone has a lateral surface area of** 10π **square feet, a slant height of** 2 **feet, and a base whose radius is** 3 **feet, what is the radius of the top of the frustum.**

We are given that $A = 10\pi$, $s = 2$, and $R = 3$, and we are asked to find r. So, using the formula we derived in part (a), we have,

$$\begin{aligned} r &= \frac{A}{\pi s} - R \\ &= \frac{10\pi}{\pi \cdot 2} - 3 \\ &= 5 - 3 \\ &= 2 \end{aligned}$$

Therefore the radius of the top of the frustum is 2 feet.

Exercise. *Identify the domain and range of the relation $y = -|1/x|$.*

This relation gives a value of y for every value of x except $x = 0$. Therefore the domain (being all possible x-coordinates) is all real numbers except zero.

The range (being the set of all possible y-coordinates) does not include positive numbers (because y is given as the negative of an absolute value), and does not include zero (since $1/x$ may get close to zero when x is very large, but never actually becomes zero). Every negative value is possible for y (by making x its reciprocal). Therefore the range is $(-\infty, 0)$.

Exercise. *Determine which of the following relations are functions of x.*

(a) $x + y^2 = 16$

This relation is not a function. For example, taking 0 as the value of x we then have $0 + y^2 = 16$ which is satisfied (made true) by $y = 4$ and also by $y = -4$. Therefore this relation associates two distinct values for y to this one value for x. (Note that this precisely means that the relation fails the Vertical Line Test.) Therefore this relation is not a function.

(b) $y - x^2 = 0$

This is equivalent to $y = x^2$ which associates exactly one value for y to each value of x. Therefore this relation is a function.

Exercise. *Let g be a function defined by $g(x) = \frac{x-1}{-x-1}$.*

(a) **Is -1 in the domain of g?**

No, -1 is not in the domain of g since $\frac{-1-1}{-(-1)-1}$ is undefined (since the denominator is zero).

(b) **Is 1 in the domain of g?**

Yes, 1 is in the domain of g since $\frac{1-1}{-1-1}$ is a real number ($g(1) = 0$).

(c) **What is the value of $g(4)$?**

We have $g(4) = \frac{4-1}{-4-1} = \frac{3}{-5} = -\frac{3}{5}$.

Exercise. *Determine whether the graphs of the functions f and g, defined below, intersect, and if they do intersect, identify the coordinates of the point of intersection.*

$$f(x) = 2x - 3$$

$$g(x) = 3x + 4$$

Suppose f and g intersect at a point (x, y). Then $f(x) = y$ and $g(x) = y$ and hence $f(x) = g(x)$. Then,

$$\begin{aligned} & 2x - 3 = 3x + 4 \\ \Longleftrightarrow & (-2x) + 2x - 3 = (-2x) + 3x + 4 \\ \Longleftrightarrow & -3 = (-2 + 3)x + 4 \\ \Longleftrightarrow & -3 = x + 4 \\ \Longleftrightarrow & -3 + (-4) = x + 4 + (-4) \\ \Longleftrightarrow & -7 = x \end{aligned}$$

Indeed, we have $f(-7) = 2(-7) - 3 = -17$ and $g(-7) = 3(-7) + 4 = -17$ so that $(-7, -17)$ is a point of intersection of the graphs of f and g. And since -7 is the set of ALL solutions of $f(x) = g(x)$, this point must be the only point of intersection. (Naturally, since f and g are straight lines, and are not the same line, they cannot intersect at more than one point).

Exercise. *Let* $A = \{x : 0 \le x < 1\}$ *and let* $B = \{x : -1 < x \le 0\}$.

(a) Find $A \cap B$.

The set $A \cap B$ is the **intersection** of A and B. Hence it contains precisely those elements of A that are also in B. Therefore $A \cap B = \{0\}$.

(b) Find $A \cup B$.

The set $A \cup B$ is the **union** of A and B. Hence it contains everything in A combined with everything in B. Therefore $A \cup B = \{x : -1 < x < 1\}$.

Exercise. *Solve* $x - 3 \le 2$ *and* $6x + 5 \ge -1$.

We have,

$$\begin{aligned} & x - 3 \le 2 \text{ and } 6x + 5 \ge -1 \\ \Longleftrightarrow & x \le 5 \text{ and } 6x \ge -6 \text{ [by adding 3 to the first and } -5 \text{ to the second]} \\ \Longleftrightarrow & x \le 5 \text{ and } x \ge -1 \text{ [by multiplying the second inequality by } \tfrac{1}{6}\text{]} \end{aligned}$$

Therefore the solution set is the intersection of the solution sets of $x \le 5$ and $x \ge -1$. Therefore the solution set is $\{x : -1 \le x \le 5\}$. In interval notation this is $[-1, 5]$.

Exercise. *Solve* $x - 2 < -4$ *or* $x + 3 > 8$.

We have,

$$\begin{aligned} & x - 2 < -4 \text{ or } x + 3 > 8 \\ \Longleftrightarrow & x < -2 \text{ or } x > 5 \text{ [by adding 2 to the first and } -3 \text{ to the second inequalities]} \end{aligned}$$

Therefore the solution set is the union of the solution set of $x < -2$ with the solution set of $x > 5$.

In interval notation this is $(-\infty, -2) \cup (5, \infty)$.

Exercise. *Solve* $-4 \leq \frac{3-4x}{-3} < 3$.

We have,

$$\begin{aligned} & -4 \leq \frac{3-4x}{-3} < 3 \\ \iff & 12 \geq 3 - 4x > -9 \text{ [by multiplying by } -3 \text{ (and hence reversing ineq. directions)]} \\ \iff & 9 \geq -4x > -12 \text{ [by adding } -3] \\ \iff & -\frac{9}{4} \leq x < 3 \text{ [by multiplying by } -\tfrac{1}{4} \text{ (and hence reversing ineq. directions)]} \end{aligned}$$

Therefore the solution set is $[-\frac{9}{4}, 3)$. (Note that $-\frac{9}{4}$ is indeed less than 3. We should always check to see that the left endpoint of an interval is less than or possibly equal to the right endpoint, else the interval would actually be empty.)

Exercise. *Solve* $|3y + 1| - 5 = -3$.

We have,

$$\begin{aligned} & |3y+1| - 5 = -3 \\ \iff & |3y+1| = 2 \text{ [by adding 5]} \\ \iff & 3y + 1 = 2 \text{ or } 3y + 1 = -2 \text{ [since } 3y+1 \text{ is 2 distant from 0.]} \\ \iff & 3y = 1 \text{ or } 3y = -3 \text{ [by adding } -1 \text{ to both]} \\ \iff & y = \frac{1}{3} \text{ or } y = -1 \text{ [by multiplying both by } \tfrac{1}{3}] \end{aligned}$$

Therefore the solution set is $\{\frac{1}{3}, -1\}$.
Checking, we see that $|3(\frac{1}{3}) + 1| - 5 = -3$ and $|3(-1) + 1| - 5 = -3$ as required.

Exercise. *Solve* $2|x - 3| + 3 < 9$.

We have,

$$\begin{aligned} & 2|x-3| + 3 < 9 \\ \iff & 2|x-3| < 6 \text{ [by adding } -3] \\ \iff & |x-3| < 3 \text{ [by multiplying by } \tfrac{1}{2}] \\ \iff & -3 < x - 3 < 3 \text{ [since } x-3 \text{ is strictly less than 3 distant from 0]} \\ \iff & 0 < x < 6 \text{ [by adding 3]} \end{aligned}$$

Therefore the solution set is the interval $(0, 6)$.

Exercise. *The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and internal diameter of the pipe and inversely with the thickness of the pipe. The stress is* 100 *pounds per square inch when the diameter is* 5 *inches, the thickness is* 0.75 *inch, and the internal pressure is* 25 *pounds per square inch. Find the* *stress when the internal pressure is* 50 *pounds per square inch, the diameter is* 6 *inches, and the thickness is* 0.5 *inch.*

Let s be the stress, d the diameter, t the thickness, and p the internal pressure. Then

$$s = \frac{kpd}{t}$$

for some constant of proportionality k.

Using the known conditions, we then have

$$\begin{aligned} 100 &= \frac{k(25)(5)}{0.75} \\ \iff 75 &= k(25)(5) \text{ [by multiplying by 0.75]} \\ \iff 3 &= k(5) \text{ [by multiplying by } \tfrac{1}{25}\text{]} \\ \iff \frac{3}{5} &= k \text{ [by multiplying by } \tfrac{1}{5}\text{]} \end{aligned}$$

Therefore $s = \frac{3pd}{5t}$ so that when the internal pressure is 50 pounds per square inch, the diameter is 6 inches, and the thickness is 0.5 inch, we then have,

$$\begin{aligned} s &= \frac{3(50)(6)}{5(0.5)} \\ &= \frac{3(50)(6)(2)}{5} \text{ [e.g. by multiplying the fraction by } \tfrac{2}{2}\text{]} \\ &= 360 \end{aligned}$$

and hence the stress is 360 pounds per square inch.

Exercise. *A thousand is a* 1 *followed by* 3 *zeros. A million is a* 1 *followed by* 6 *zeros. A google is a* 1 *followed by* 100 *zeros. Express a google in scientific notation.*

A google in scientific notation is 1×10^{100}.

Exercise. *Exercise.*

(a) Determine the coefficient and degree of $-8x^2y^3$.
Coefficient is -8. Degree is 5.
(b) Is $2x^{-1} + 3x$ a polynomial?
No.
(c) Is $5x^2 - 9x + 1$ a polynomial?
Yes.
(d) Is $\frac{4}{z-1}$ a polynomial?
No.
(e) Is $4mn^3 - 2m^2n^3 + mn^8$ a polynomial?
Yes.

Exercise. *Simplify* $(x^2 + 5x + 1) + (3x^2 - 2x - 3)$.

This is simply $4x^2 + 3x - 2$.

Exercise. *Let* $f(x) := -2x^3 + 3x - 1$ *and let* $g(x) := 2x^2 + 3$.

(a) What is $f(-3)$?
$f(-3) = -2(-3)^3 + 3(-3) - 1 = -2(-27) - 9 - 1 = 54 - 10 = 44$
(b) What is $(f + g)(x)$?
$(f + g)(x) = f(x) + g(x) = (-2x^3 + 3x - 1) + (2x^2 + 3) = -2x^3 + 2x^2 + 3x + 2$
(c) What is $(f - g)(1)$?
$(f - g)(1) = f(1) - g(1) = (-2 + 3 - 1) - (2 + 3) = -5$

Exercise. *Multiply and simplify* $(4a + 3b)(a - 5b)$.

We have,

$$\begin{aligned}(4a + 3b)(a - 5b) &= (4a)(a) + (4a)(-5b) + (3b)(a) + (3b)(-5b) \\ &= 4a^2 - 20ab + 3ab - 15b^2 \\ &= 4a^2 - 17ab - 15b^2\end{aligned}$$

Exercise. *Divide*

$$\frac{4m^2n^2 + 6m^2n - 18mn^2}{4m^2n^2}$$

.

We have,

$$\begin{aligned}\frac{4m^2n^2 + 6m^2n - 18mn^2}{4m^2n^2} &= \frac{4m^2n^2}{4m^2n^2} + \frac{6m^2n}{4m^2n^2} - \frac{18mn^2}{4m^2n^2} \\ &= 1 + \frac{3}{2n} - \frac{9}{2m}\end{aligned}$$

Exercise. *Let* $f(x) := 5x^3 + 8x^2 - 7x - 6$. *Is* $(x + 2)$ *a factor of* $f(x)$*? (Justify your answer.)*

By the Factor Theorem, $(x + 2)$ is a factor of $f(x)$ if and only if $f(-2) = 0$. (Since being a factor would mean $f(x)$ is $(x + 2)$ times something, and then $f(-2)$ would be $(-2 + 2)$ times something so that $f(x)$ would be zero.)

In this case we have

$$f(-2) = 5(-2)^3 + 8(-2)^2 - 7(-2) - 6 = 5(-8) + 8(4) + 14 - 6 = -40 + 32 + 8 = 0$$

and therefore $(x + 2)$ is indeed a factor of $f(x)$.

Exercise. *Let* $f(x) := x^4 - 1$. *What is the remainder when* $f(x)$ *is divided by* $x - 1$*?*

By the Remainder Theorem, the remainder when dividing $f(x)$ by $x - 1$ is $f(1)$. Therefore the remainder is $f(1) = 1^4 - 1 = 0$.

Exercise. *Solve* $2q^2 + 3q - 14 = 0$.

Factoring the left side, we have,

$$\begin{aligned}
&2q^2 + 3q - 14 = 0 \\
\iff &(2q+7)(q-2) = 0 \\
\iff &2q + 7 = 0 \text{ or } q - 2 = 0 \\
\iff &2q = -7 \text{ or } q = 2 \\
\iff &q = -\frac{7}{2} \text{ or } q = 2
\end{aligned}$$

Therefore the solution set is $\{-\frac{7}{2}, 2\}$.
Checking, we see that

$$\begin{aligned}
2\left(-\frac{7}{2}\right)^2 + 3\left(-\frac{7}{2}\right) - 14 &= 2\left(\frac{49}{4}\right) - \frac{21}{2} - 14 \\
&= \frac{49}{2} - \frac{21}{2} - 14 \\
&= \frac{49-21}{2} - 14 \\
&= \frac{21}{2} - 14 \\
&= 14 - 14 \\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
2(2)^2 + 3(2) - 14 &= 2(4) + 6 - 14 \\
&= 8 + 6 - 14 \\
&= 0
\end{aligned}$$

So that these both are indeed solutions.

Exercise. *Factor completely* $2p^4q + 14p^3q - 32p^2q - 224pq$.

We have,

$$\begin{aligned}
&2p^4q + 14p^3q - 32p^2q - 224pq \\
=&2pq(p^3 + 7p^2 - 16p - 112) \text{ [e.g. by factoring out the GCF of all the terms]} \\
=&2pq[(p^3 + 7p^2) + (-16p - 112)] \text{ [trying our luck with some grouping]} \\
=&2pq[p^2(p+7) - 16(p+7)] \text{ [e.g. by factoring each of these groups]} \\
=&2pq(p+7)(p^2 - 16) \text{ [e.g. by factoring out the GCF of the terms]} \\
=&2pq(p+7)(p+4)(p-4) \text{ [noting the Difference of Squares]}
\end{aligned}$$

Exercise. *Factor* $24r^2 + 23rs - 12s^2$.

We should try the following possible forms,

$$\begin{gathered}(r + \boxed{?}s)(24r + \boxed{?}s)\\(2r + \boxed{?}s)(12r + \boxed{?}s)\\(3r + \boxed{?}s)(8r + \boxed{?}s)\\(4r + \boxed{?}s)(6r + \boxed{?}s)\end{gathered}$$

filling in pairs of factors for -12.

By this method, with perseverance we find that

$$\begin{aligned}(3r + 4s)(8r - 3s) &= 24r^2 - 9rs + 32rs - 12s^2\\&= 24r^2 + 23rs - 12s^2\end{aligned}$$

as desired.

Therefore the factorization of $24r^2 + 23rs - 12s^2$ is $(3r + 4s)(8r - 3s)$.

Exercise. *Factor* $64x^3y - 125y$.

We have,

$$\begin{aligned}&64x^3y - 125y\\=&y(64x^3 - 125) \text{ [by factoring out the GCF (always do this first!)]}\\=&y[(4x)^3 - (5)^3] \text{ [revealing the Difference of Cubes]}\\=&y(4x - 5)[(4x)^2 + (4x)(5) + (5)^2] \text{ [applying } a^3 - b^3 = (a - b)(a^2 + ab + b^2)\text{]}\\=&y(4x - 5)(16x^2 + 20x + 25)\end{aligned}$$

Now we should try to factor $(16x^2 + 20x + 25)$ using the following possible forms,

$$\begin{gathered}(x + \boxed{?})(16r + \boxed{?})\\(2x + \boxed{?})(8r + \boxed{?})\\(4x + \boxed{?})(4r + \boxed{?})\end{gathered}$$

filling in pairs of factors for 25.

But in this way we find that no combination works and hence $(16x^2 + 20x + 25)$ is prime (over rational coefficients).

Therefore our factorization,

$$64x^3y - 125y = y(4x - 5)(16x^2 + 20x + 25)$$

is complete.

Exercise. *Simplify*

$$\frac{(2ab^2c)^{-1}}{(a^{-1}b^3c^2)^{-2}}$$

.

We have,

$$\begin{aligned}
&\frac{(2ab^2c)^{-1}}{(a^{-1}b^3c^2)^{-2}}\\
=&\frac{(a^{-1}b^3c^2)^2}{(2ab^2c)^1}\\
=&\frac{(a^{-1})^2(b^3)^2(c^2)^2}{2ab^2c}\\
=&\frac{a^{-2}b^6c^4}{2ab^2c}\\
=&2^{-1}a^{-2-1}b^{6-2}c^{4-1}\\
=&2^{-1}a^{-3}b^4c^3\\
=&\frac{b^4c^3}{2a^3}
\end{aligned}$$

Exercise. *A cannonball is fired from a cliff that is* 260 *feet high with an initial speed of* 128 *feet per second. The height* s *of the cannonball (in feet) as a function of time (in seconds) can be modeled by the function*

$$s(t) = -16t^2 + 64t + 260$$

When will the height of the cannonball be 320 feet?

Since we are given the height s as a function of time, we need only solve the equation $s(t) = 320$ for t.

So,

$$\begin{aligned}
&s(t) = 320\\
\iff& -16t^2 + 64t + 260 = 320\\
\iff& -16t^2 + 64t + 260 - 320 = 0\\
\iff& -16t^2 + 64t - 60 = 0\\
\iff& 4t^2 - 16t + 15 = 0 \text{ [by multiplying by } -\tfrac{1}{4}]\\
\iff& (2t-3)(2t-5) = 0 \text{ [by factoring the left side]}\\
\iff& 2t - 3 = 0 \text{ or } 2t - 5 = 0\\
\iff& 2t = 3 \text{ or } 2t = 5\\
\iff& t = \frac{3}{2} \text{ or } t = \frac{5}{2}
\end{aligned}$$

Therefore $s(t) = 320$ precisely when $t = \frac{3}{2}$ or $t = \frac{5}{2}$ (it happens twice).

We are more accustomed to giving time in decimals rather than fractions, so we should say the height of the cannonball will be 320 feet at either 1.5 seconds or again at 2.5 seconds (presumably after being fired from the cannon).

Exercise. *Let* $A = \{x : 1 \leq x < 3\}$ *and let* $B = \{x : 2 < x \leq 4\}$.

(a) Find $A \cap B$.

The set $A \cap B$ is the **intersection** of A and B. Hence it contains precisely those elements of A that are also in B. Therefore $A \cap B = \{x : 2 < x < 3\}$. Or we can write this in interval notation as $(2, 3)$.

(b) Find $A \cup B$.

The set $A \cup B$ is the **union** of A and B. Hence it contains everything in A combined with everything in B. Therefore $A \cup B = \{x : 1 \leq x \leq 4\}$. Or we can write this in interval notation as $[1, 4]$.

Exercise. *Solve:* $y - 3 > 2$ *or* $y - 2 < -6$.

We have,

$$\begin{aligned} & y - 3 > 2 \text{ or } y - 2 < -6 \\ \iff & y > 5 \text{ or } y < -4 \text{ [by adding 3 to the first and 2 to the second]} \end{aligned}$$

Therefore the solution set is the union of the solution set of $y > 5$ with the solution set of $y < -4$.

In interval notation this is $(-\infty, -4) \cup (5, \infty)$.

Exercise. *Solve:* $7x + 9 \geq 2$ *and* $x - 4 \leq 1$.

We have,

$$\begin{aligned} & 7x + 9 \geq 2 \text{ and } x - 4 \leq 1 \\ \iff & 7x \geq -7 \text{ and } x \leq 5 \text{ [by adding } -9 \text{ to the first and 4 to the second]} \\ \iff & x \geq -1 \text{ and } x \leq 5 \text{ [by multiplying the first inequality by } \tfrac{1}{7}] \end{aligned}$$

Therefore the solution set is the intersection of the solution sets of $x \geq -1$ and $x \leq 5$. Therefore the solution set is $\{x : -1 \leq x \leq 5\}$. In interval notation this is $[-1, 5]$.

Exercise. *Solve* $-4 < \frac{3-2x}{3} \leq 3$.

We have,

$$\begin{aligned} & -4 < \frac{3-2x}{3} \leq 3 \\ \iff & -12 < 3 - 2x \leq 9 \text{ [by multiplying by 3]} \\ \iff & -15 < -2x \leq 6 \text{ [by adding } -3] \\ \iff & \frac{15}{2} > x \geq -3 \text{ [by multiplying by } -\tfrac{1}{2} \text{ (and hence reversing ineq. directions)]} \\ \iff & -3 \leq x < \frac{15}{2} \end{aligned}$$

Therefore the solution set is $[-3, \frac{15}{2})$. (Note that -3 is indeed less than $\frac{15}{2}$. We should always check to see that the left endpoint of an interval is less than or possibly equal to the right endpoint, else the interval would actually be empty.)

Exercise. *Solve* $|2x+3|-7=-2$.

We have,

$$\begin{aligned}
&|2x+3|-7=-2\\
\iff&|2x+3|=5 \text{ [by adding 7]}\\
\iff&2x+3=5 \text{ or } 2x+3=-5 \text{ [since } 2x+3 \text{ is 5 distant from 0.]}\\
\iff&2x=2 \text{ or } 2x=-8 \text{ [by adding } -3 \text{ to both]}\\
\iff&x=1 \text{ or } x=-4 \text{ [by multiplying both by } \tfrac{1}{2}\text{]}
\end{aligned}$$

Therefore the solution set is $\{-4,1\}$.
Checking, we see that $|2(-4)+3|-7=-2$ and $|2(1)+3|-7=-2$ as required.

Exercise. *Solve* $-2|x-3|+3>-9$.

We have,

$$\begin{aligned}
&-2|x-3|+3>-9\\
\iff&-2|x-3|>-12 \text{ [by adding } -3\text{]}\\
\iff&|x-3|<6 \text{ [by multiplying by } -\tfrac{1}{2} \text{ (and hence reversing ineq. directions)]}\\
\iff&-6<x-3<6 \text{ [since } x-3 \text{ is strictly less than 6 distant from 0]}\\
\iff&-3<x<9 \text{ [by adding 3]}
\end{aligned}$$

Therefore the solution set is the interval $(-3,9)$.

Exercise. *The stress in the material of a pipe subject to internal pressure varies jointly with the internal pressure and internal diameter of the pipe and inversely with the thickness of the pipe. The stress is* 100 *pounds per square inch when the diameter is* 5 *inches, the thickness is* 0.75 *inch, and the internal pressure is* 25 *pounds per square inch. Find the stress when the internal pressure is* 25 *pounds per square inch, the diameter is* 6 *inches, and the thickness is* 1 *inch.*

Let s be the stress, d the diameter, t the thickness, and p the internal pressure. Then

$$s=\frac{kpd}{t}$$

for some constant of proportionality k.

Using the known conditions, we then have

$$\begin{aligned}
100&=\frac{k(25)(5)}{0.75}\\
\iff 75&=k(25)(5) \text{ [by multiplying by 0.75]}\\
\iff 3&=k(5) \text{ [by multiplying by } \tfrac{1}{25}\text{]}\\
\iff \frac{3}{5}&=k \text{ [by multiplying by } \tfrac{1}{5}\text{]}
\end{aligned}$$

Therefore $s = \frac{3pd}{5t}$ so that when the internal pressure is 25 pounds per square inch, the diameter is 6 inches, and the thickness is 1 inch, we then have,

$$\begin{aligned} s &= \frac{3(25)(6)}{5(1)} \\ &= 3(5)(6) \\ &= 90 \end{aligned}$$

and hence the stress is then 90 pounds per square inch.

Exercise. *Consider the system*

$$\begin{cases} 3x + y = 1 \\ 6x - 2y = 10 \end{cases}$$

(a) Solve the system.

By multiplying the second equation by $\frac{1}{2}$ we see that this system is equivalent to:

$$\begin{cases} 3x + y = 1 \\ 3x - y = 5 \end{cases}$$

Then by adding these two equations together (the Elimination Method) we get $6x = 6$ so that $x = 1$.

Therefore $3(1) + y = 1$ [by substituting the value of x into the first equation (but we could just as well have used the second equation)] and hence $y = -2$.

Therefore the solution set is $\{(1, -2)\}$.

(b) Is this system consistent or inconsistent? If consistent, is it independent or dependent?

This system is **consistent** since it has solutions, and **independent** since the solution set is not the same as the solution set of either equation.

Exercise. *Consider the system*

$$\begin{cases} x - 2y = -2 \\ x - 2y = 2 \end{cases}$$

(a) Solve the system.

The solution set is $\{\}$ (the empty set) since it is impossible to satisfy both equations. (We can use the Substitution Method or Elimination Method. Either way we will reach a contradiction).

(b) Is this system consistent or inconsistent? If consistent, is it independent or dependent?

This system is **inconsistent** since it has no solutions.

Exercise. *A trillion is a* 1 *followed by* 12 *zeros. Express a trillion in scientific notation.*

A trillion in scientific notation is 1×10^{12}.

Exercise.

(a) Determine the coefficient and degree of $-13a^4b^7$.
Coefficient is -13. Degree is 11.
(b) Is $y^2 - 9y + 1$ a polynomial?
Yes.
(c) Is $3x^{-2} + 7y$ a polynomial?
No. (Variables in Polynomials must have non-negative, integral powers).
(d) Is $3ab^4 - 2a^3b + ab^7$ a polynomial?
Yes.
(e) Is $\frac{7}{m+2}$ a polynomial?
No.

Exercise. *Simplify* $(x^2 - 3x + 1) + (4x^2 + x - 4)$.

This is simply $5x^2 - 2x - 3$.

Exercise. *Let* $f(x) := -2x^3 + 4x - 2$ *and let* $g(x) := x^2 + 4$.

(a) What is $f(-2)$?
$f(-2) = -2(-2)^3 + 4(-2) - 2 = -2(-8) - 8 - 2 = 16 - 10 = 6$
(b) What is $(f - g)(1)$?
$(f - g)(1) = f(1) - g(1) = (-2 + 4 - 2) - (1 + 4) = -5$
(c) What is $(f + g)(x)$?
$(f + g)(x) = f(x) + g(x) = (-2x^3 + 4x - 2) + (x^2 + 4) = -2x^3 + x^2 + 4x + 2$

Exercise. *Multiply and simplify* $(3m + 4n)(2m - n)$.

$$\begin{aligned}(3m + 4n)(2m - n) &= 6m^2 - 3mn + 8mn - 4n^2 \\ &= 6m^2 + 5mn - 4n^2\end{aligned}$$

Exercise. *Let* $f(x) := 5x^3 + 8x^2 - 7x - 6$. *Is* $(x + 2)$ *a factor of* $f(x)$*? (Justify your answer.)*

By the Factor Theorem, $(x + 2)$ is a factor of $f(x)$ if and only if $f(-2) = 0$. (Since being a factor would mean $f(x)$ is $(x + 2)$ times something, and then $f(-2)$ would be $(-2 + 2)$ times something so that $f(x)$ would be zero.)

In this case we have

$$f(-2) = 5(-2)^3 + 8(-2)^2 - 7(-2) - 6 = 5(-8) + 8(4) + 14 - 6 = -40 + 32 + 8 = 0$$

and therefore $(x + 2)$ is indeed a factor of $f(x)$.

Exercise. *Solve* $2p^2 - 7p - 15 = 0$.

Factoring the left side, we have,

$$\begin{aligned} &2p^2 - 7p - 15 = 0 \\ \Longleftrightarrow &(2p+3)(p-5) = 0 \\ \Longleftrightarrow &2p+3 = 0 \text{ or } p-5 = 0 \\ \Longleftrightarrow &2p = -3 \text{ or } p = 5 \\ \Longleftrightarrow &p = -\frac{3}{2} \text{ or } p = 5 \end{aligned}$$

Therefore the solution set is $\{-\frac{3}{2}, 5\}$.

Exercise. *Factor completely* $2x^5y - 18x^3y + 12x^4y - 108x^2y$.

We have,

$$\begin{aligned} &2x^5y - 18x^3y + 12x^4y - 108x^2y \\ =&2x^2y(x^3 - 9x + 6x^2 - 54) \text{ [e.g. by factoring out the GCF of all the terms]} \\ =&2x^2y[(x^3 - 9x) + (6x^2 - 54)] \text{ [trying our luck with some grouping]} \\ =&2x^2y[x(x^2 - 9) + 6(x^2 - 9)] \text{ [e.g. by factoring each of these groups]} \\ =&2x^2y(x^2 - 9)(x + 6) \text{ [e.g. by factoring out the GCF of the terms]} \\ =&2x^2y(x + 3)(x - 3)(x + 6) \text{ [by factoring the Difference of Squares]} \end{aligned}$$

Exercise. *Factor* $64x - 27xy^3$.

We have,

$$\begin{aligned} &64x - 27xy^3 \\ =&x(64 - 27y^3) \text{ [by factoring out the GCF (always do this first!)]} \\ =&x[4^3 - (3y)^3] \text{ [revealing the Difference of Cubes]} \\ =&x(4 - 3y)[(4)^2 + (4)(3y) + (3y)^2] \text{ [applying } a^3 - b^3 = (a-b)(a^2 + ab + b^2)\text{]} \\ =&x(4 - 3y)(16 + 12y + 9y^2) \\ =&x(4 - 3y)(9y^2 + 12y + 16) \text{ [ordering the trinomial factor by decreasing powers of } y\text{]} \end{aligned}$$

Now we should try to factor $(9y^2 + 12y + 16)$ using the following possible forms,

$$\begin{aligned} (y + \boxed{?})(9y + \boxed{?}) \\ (3y + \boxed{?})(3y + \boxed{?}) \end{aligned}$$

filling in pairs of factors for 16.

But in this way we find that no combination works and hence $(9y^2 + 12y + 16)$ is prime (over rational coefficients).

Therefore our factorization,

$$64x - 27xy^3 = x(4 - 3y)(9y^2 + 12y + 16)$$

is complete.

Exercise. *Simplify*

$$\frac{(2abc^2)^{-2}}{(a^3b^{-1}c)^{-1}}$$

.

We have,

$$\begin{aligned}
\frac{(2abc^2)^{-2}}{(a^3b^{-1}c)^{-1}} &= \frac{(a^3b^{-1}c)^1}{(2abc^2)^2} \text{ [since } A^{-n} = \tfrac{1}{A^n}\text{]} \\
&= \frac{a^3b^{-1}c}{2^2a^2b^2(c^2)^2} \text{ [since } A^1 = A \text{ and } (AB)^n = A^nB^n\text{]} \\
&= \frac{a^3b^{-1}c}{4a^2b^2c^4} \text{ [since } (A^n)^m = A^{nm}\text{]} \\
&= 4^{-1}a^{3-2}b^{-1-2}c^{1-4} \text{ [since } \tfrac{A^n}{A^m} = A^{n-m}\text{]} \\
&= 4^{-1}ab^{-3}c^{-3} \\
&= \frac{a}{4b^3c^3}
\end{aligned}$$

Exercise. *Factor completely* $x^{12} - 1$.

We have,

$$\begin{aligned}
x^{12} - 1 &= (x^6)^2 - 1^2 \text{ [revealing the Difference of Squares.]} \\
&= (x^6+1)(x^6-1) \text{ [by applying the } A^2 - B^2 = (A+B)(A-B) \text{ formula.]} \\
&= [(x^2)^3 + 1^3][(x^2)^3 - 1^3] \text{ [revealing the Sum and the Difference of Cubes.]} \\
&= (x^2+1)[(x^2)^2 - x^2 + 1^2](x^2-1)[(x^2)^2 + x^2 + 1^2] \text{ [by applying the formulas.]} \\
&= (x^2+1)(x^4-x^2+1)(x+1)(x-1)(x^4+x^2+1)
\end{aligned}$$

by simplifying and factoring another Difference of Squares.

Attempts to factor the Trinomials reveal that they are prime. Therefore this is a complete factorization of $x^{12} - 1$.