Algebra

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We will make frequent use of the following two theorems in the problem that follows:

2.2 Theorem. *Every finitely generated abelian group G is (isomorphic to) a finite direct sum of cyclic groups, each of which is either infinite or of order a power of a prime.*

2.3 Lemma. *If m is a positive integer and $m = p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$ ($p_1, \ldots, p_t$ distinct primes and each $n_i > 0$), then $\mathbf{Z}_m \cong \mathbf{Z}_{p_1^{n_1}} \oplus \mathbf{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbf{Z}_{p_t^{n_t}}$.*

p.82, #12 Problem.

(a) What are the elementary divisors of the group $\mathbf{Z}_2 \oplus \mathbf{Z}_9 \oplus \mathbf{Z}_{35}$; what are its invariant factors? Do the same for $\mathbf{Z}_{26} \oplus \mathbf{Z}_{42} \oplus \mathbf{Z}_{49} \oplus \mathbf{Z}_{200} \oplus \mathbf{Z}_{1000}$.

Let $G = \mathbf{Z}_2 \oplus \mathbf{Z}_9 \oplus \mathbf{Z}_{35}$.

Then $G \cong \mathbf{Z}_2 \oplus \mathbf{Z}_{3^2} \oplus (\mathbf{Z}_5 \oplus \mathbf{Z}_7)$ by Lemma 2.3 since 5 and 7 are mutually prime.

Therefore the elementary divisors of G are $\boxed{2, 5, 7, 3^2}$.

$G \cong \mathbf{Z}_{630}$ by Lemma 2.3 since 2, 9, and 35 are pairwise mutually prime. Therefore the invariant factors are $\boxed{630}$.

Let $H = \mathbf{Z}_{26} \oplus \mathbf{Z}_{42} \oplus \mathbf{Z}_{49} \oplus \mathbf{Z}_{200} \oplus \mathbf{Z}_{1000}$.

Then $H \cong (\mathbf{Z}_2 \oplus \mathbf{Z}_{13}) \oplus (\mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_7) \oplus \mathbf{Z}_{7^2} \oplus (\mathbf{Z}_{2^3} \oplus \mathbf{Z}_{5^2}) \oplus (\mathbf{Z}_{2^3} \oplus \mathbf{Z}_{5^3})$.

Therefore the elementary divisors are $\boxed{2, 2, 2^3, 2^3, 3, 5^2, 5^3, 7, 7^2, 13}$.

Now, taking these divisors in mutually prime groups such that the product of each divides the product of the last, we have

$$\begin{aligned} 2 &= 2 \\ 2 &= 2 \\ 2^3 \cdot 5^2 \cdot 7 &= 1400 \\ 2^3 \cdot 3 \cdot 5^3 \cdot 7^2 \cdot 13 &= 1911000 \end{aligned}$$

so that $H \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_{1400} \oplus \mathbf{Z}_{1911000}$.

Therefore the invariant factors are $\boxed{2, 2, 1400, 1911000}$.

(b) Determine up to isomorphism all abelian groups of order 64*; do the same for order 96.*

We have $64 = 2^6$ so that the possible families of elementary divisors are

$$\{2,2,2,2,2,2\}$$
$$\{2^2,2,2,2,2\}$$
$$\{2^3,2,2,2\}$$
$$\{2^4,2,2\}$$
$$\{2^5,2\}$$
$$\{2^6\}$$
$$\{2^2,2^2,2,2\}$$
$$\{2^2,2^3,2\}$$
$$\{2^2,2^4\}$$
$$\{2^3,2^3\}$$
$$\{2^2,2^2,2^2\}$$

Therefore by Theorem 2.2 all abelian groups of order 64 are isomorphic to one of the following:

$$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^3} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^4} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^5} \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^6}$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^2} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^3} \oplus \mathbf{Z}_2$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^4}$$
$$\mathbf{Z}_{2^3} \oplus \mathbf{Z}_{2^3}$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^2}$$

For order 96 we have, $96 = 2^5 \cdot 3$ so that the possible families of elementary divisors are:

$$\{2,2,2,2,2,3\}$$
$$\{2^2,2,2,2,3\}$$
$$\{2^3,2,2,3\}$$
$$\{2^4,2,3\}$$
$$\{2^5,3\}$$
$$\{2^2,2^2,2,3\}$$
$$\{2^2,2^3,3\}$$

Therefore by Theorem 2.2 all abelian groups of order 96 are isomorphic to one of the following:

$$\mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^3} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^4} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^5} \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^2} \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3$$
$$\mathbf{Z}_{2^2} \oplus \mathbf{Z}_{2^3} \oplus \mathbf{Z}_3$$

(c) Determine all abelian groups of order n for $n \leq 20$.

$$
\begin{aligned}
n = 1: &\quad \mathbf{Z}_1 \\
n = 2: &\quad \mathbf{Z}_2 \\
n = 3: &\quad \mathbf{Z}_3 \\
n = 4: &\quad \mathbf{Z}_4, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\
n = 5: &\quad \mathbf{Z}_5 \\
n = 6: &\quad \mathbf{Z}_6 \\
n = 7: &\quad \mathbf{Z}_7 \\
n = 8: &\quad \mathbf{Z}_8, \quad \mathbf{Z}_4 \oplus \mathbf{Z}_2, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\
n = 9: &\quad \mathbf{Z}_9, \quad \mathbf{Z}_3 \oplus \mathbf{Z}_3 \\
n = 10: &\quad \mathbf{Z}_{10} \\
n = 11: &\quad \mathbf{Z}_{11} \\
n = 12: &\quad \mathbf{Z}_{12}, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_3 \\
n = 13: &\quad \mathbf{Z}_{13} \\
n = 14: &\quad \mathbf{Z}_{14} \\
n = 15: &\quad \mathbf{Z}_{15} \\
n = 16: &\quad \mathbf{Z}_{16}, \quad \mathbf{Z}_8 \oplus \mathbf{Z}_2, \quad \mathbf{Z}_4 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2, \quad \mathbf{Z}_4 \oplus \mathbf{Z}_4, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2 \\
n = 17: &\quad \mathbf{Z}_{17} \\
n = 18: &\quad \mathbf{Z}_{18}, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_3 \oplus \mathbf{Z}_3 \\
n = 19: &\quad \mathbf{Z}_{19} \\
n = 20: &\quad \mathbf{Z}_{20}, \quad \mathbf{Z}_2 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_5
\end{aligned}
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