Real Analysis: Exam III

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1. Problem. *For a function $f : A \to \mathbf{R}$, fully explain the similarities and differences of the following statements:*

(i) The limit of f at c is L. Both L and c are real numbers.

(ii) f is continuous at c. c is a real number.

We will consider the conditions under which the truth or falsity of statements (i) and (ii) correspond or differ for a given function $f : A \to \mathbf{R}$ and a given real number c according to the following two main cases:

(a) when c is a cluster point of A, and

(b) when c is not a cluster point of A.

When c is a cluster point of A, meaning for every $\delta > 0$ there exists at least one point $x \in A, x \neq c$ within the δ-neighborhood of c, then the limit of f at c may or may not exist. If the limit does not exist, i.e. when statement (i) is false, then statement (ii) is false as well since continuity at c would absurdly imply that $\lim_{x\to c} f(x) = f(c)$. Otherwise, if the limit of f at c does exist, so that statement (i) is true, then statement (ii) becomes logically equivalent to the statement $f(c) = \lim_{x\to c} f(x)$. To show that this statement is equivalent to statement (ii), we will show that each implies the other. If $f(c) = \lim_{x\to c} f(x)$ then there exists a δ such that if x is in the δ-neighborhood of c but $x \neq c$ then $|f(x) - f(c)| < \epsilon$ for any $\epsilon > 0$. The definition of continuity also requires, when $c \in A$, that $|f(x) - f(c)| < \epsilon$ if $x = c$, but this is trivially satisfied in this case by $|f(c) - f(c)| = 0$. Conversely, if f is continuous at c then there exists a δ such that if x is in the δ-neighborhood of c then $|f(x) - f(c)| < \epsilon$ for any $\epsilon > 0$ and hence $\lim_{x\to c} f(x) = f(c)$. Therefore if statement (i) is true, then statement (ii) is logically equivalent to the statement $f(c) = \lim_{x\to c} f(x)$.

On the other hand, when c is not a cluster point of A, then the limit of f at c by definition cannot exist. Hence in this case statement (i) is false, but statement (ii) becomes logically equivalent to the statement that f is defined at c. That f may be defined at c and yet c not be a cluster point of A follows when given a sufficiently small $\delta > 0$ there does not exist an $x \in A$ other than $x = c$ within the δ-neighborhood of c. But the existence of precisely this δ guarantees that only at $x = c$ must we have $|f(x) - f(c)| < \epsilon$ for arbitrary $\epsilon > 0$ in order to establish continuity. Hence, since $|f(c) - f(c)| = 0 < \epsilon$, continuity follows automatically. Conversely, if f is not defined at c then $|f(x) - f(c)|$ is not defined. Hence we have continuity at c if and only if f is defined at c.

The foregoing is summarized by the following table:

c is a cluster point of A	$\lim_{x\to c} f(x)$ exists	f is continuous at c
True	False	False
True	True	iff $f(c) = \lim_{x\to c} f(x)$
False	False	iff f is defined at c

3. Problem. *Provide an example that illustrates that we can claim a function f, $f : A \to \mathbf{R}$, is continuous at a real number c but the function does not have a limit at c.*

Let $f : A \to \mathbf{R}$ where $A = \{c\}$. We need to show that both of the following are true:

(i) f is continuous at c

(ii) $\lim_{x \to c} f(x)$ does not exist.

We proceed as follows.

(i) To show that f is continuous at c, we show that for any $\epsilon > 0$, $\exists \delta > 0$ such that for all $x \in A$ satisfying $|x - c| < \delta$, $|f(x) - f(c)| < \epsilon$. Suppose $\epsilon > 0$. Then for any $\delta > 0$,

$$\begin{aligned} x \in A &\Longrightarrow x = c \\ &\Longrightarrow |x - c| = 0 < \delta \\ &\Longrightarrow |f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon \end{aligned}$$

Therefore any $\delta > 0$ satisfies the conditions for any $\epsilon > 0$. Therefore f is continuous at c.

(ii) To show that $\lim_{x \to c} f(x)$ does not exist, we show that c is not a cluster point of A. Then it follows by definition that $\lim_{x \to c} f(x)$ cannot exist. To show that c is not a cluster point of A, it is sufficient to show that there does not exist a point $x \in A$ such that $x \neq c$. But since c is the only element of A, $x \in A \Longrightarrow x = c$ so that there does not exist a point $x \in A$ such that $x \neq c$. Therefore c is not a cluster point of A. Therefore $\lim_{x \to c} f(x)$ does not exist.

Therefore $f : A \to \mathbf{R}$ where $A = \{c\}$ is an example of a function that is continuous at a real number c but that does not have a limit at c. **Q.E.D.**

4. Problem. *Prove, using the definition, that* $\lim\limits_{x \to 0} \dfrac{x+1}{x^2+2} = \dfrac{1}{2}$.

Let $f : \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{x+1}{x^2+2}$. We need to show that $\lim_{x \to 0} f(x) = 1/2$. To do this, we will show that both of the following are true:

(i) 0 is a cluster point of $\mathbf{R}$.

(ii) $\forall \epsilon > 0, \exists \delta > 0$ such that if $0 < |x| < \delta$ then $|f(x) - 1/2| < \epsilon$.

We thus proceed:

(i) It is an immediate consequence of the Density Theorem that 0 is a cluster point of $\mathbf{R}$.

(ii) It remains to show that $\forall \epsilon > 0, \exists \delta > 0$ such that $0 < |x| < \delta \Longrightarrow |f(x) - 1/2| < \epsilon$.

First note that

$$\begin{aligned}\left|f(x)-\frac{1}{2}\right| &= \left|\frac{x+1}{x^2+2}-\frac{1}{2}\right| \\ &= \left|\frac{2(x+1)-(x^2+2)}{2(x^2+2)}\right| \\ &= \left|\frac{2x+2-x^2-2}{2(x^2+2)}\right| \\ &= \left|\frac{x(2-x)}{2(x^2+2)}\right| \\ &= |x|\cdot\left|\frac{2-x}{2(x^2+2)}\right|\end{aligned}$$

Suppose $\epsilon > 0$. Now we may choose $\delta = \inf\{1, \frac{4}{3}\epsilon\}$, so that for $0 < |x| < \delta$ we have

$$\begin{aligned}0<|x|<1 &\Longrightarrow 0<x^2<1 \\ &\Longrightarrow 2<x^2+2<3 \\ &\Longrightarrow 4<2(x^2+2)<6 \\ &\Longrightarrow \frac{1}{4}>\frac{1}{2(x^2+2)}>\frac{1}{6}\end{aligned}$$

and

$$\begin{aligned}0<|x|<1 &\Longrightarrow -1<x<1 \\ &\Longrightarrow 1>-x>-1 \\ &\Longrightarrow 3>2-x>1\end{aligned}$$

and then by multiplying the above two results together we have

$$\frac{3}{4}>\frac{2-x}{2(x^2+2)}>\frac{1}{6}\Longrightarrow\left|\frac{2-x}{2(x^2+2)}\right|<\frac{3}{4}$$

Hence

$$\begin{aligned}0<|x|<\delta &\Longrightarrow |x|<\frac{4}{3}\epsilon \\ &\Longrightarrow |x|\cdot\left|\frac{2-x}{2(x^2+2)}\right|<\left(\frac{4}{3}\epsilon\right)\left(\frac{3}{4}\right) \\ &\Longrightarrow |f(x)-1/2|<\epsilon\end{aligned}$$

Therefore $\lim_{x\to 0}\frac{x+1}{x^2+2}=\frac{1}{2}$. **Q.E.D.**

5. Problem. *Let $g : \mathbf{R} \to \mathbf{R}$ satisfy the relation $g(x+y) = g(x)g(y)$ for all $x, y \in \mathbf{R}$.*
(a) Show that if we have $g(a) = 0$ for some $a \in \mathbf{R}$, then $g(x) = 0$ for all $x \in \mathbf{R}$.
(b) Show that if g is continuous at $x = 0$, then g is continuous at every point of $\mathbf{R}$.

(a) Suppose $a \in \mathbf{R}$ and $g(a) = 0$. Then for any $x \in \mathbf{R}$,

$$\begin{aligned} g(x) &= g(a + -a + x) \\ &= g(a)g(-a+x) \\ &= 0g(-a+x) \\ &= 0 \end{aligned}$$

Therefore if $g(a) = 0$ for some $a \in \mathbf{R}$, then $g(x) = 0$ for all $x \in \mathbf{R}$. **Q.E.D.**

(b) Suppose g is continuous at $x = 0$, and suppose $c \in \mathbf{R}$. We need to show that g is continuous at c and hence, since c is arbitrary in $\mathbf{R}$, g is continuous at every point of $\mathbf{R}$. To do this, we will show that for any $\epsilon > 0$, $\exists \delta > 0$ such that

$$y \in \mathbf{R}, |y - c| < \delta \Longrightarrow |g(y) - g(c)| < \epsilon$$

Since g is continuous at $x = 0$, we have that for any $\epsilon_0 > 0$, $\exists \delta_0 > 0$ such that

$$\begin{aligned} |x| < \delta_0 &\Longrightarrow |g(x) - g(0)| < \epsilon_0 \\ &\Longrightarrow |g(x + c + -c) - g(c + -c)| < \epsilon_0 \\ &\Longrightarrow |g(x+c)g(-c) - g(c)g(-c)| < \epsilon_0 \\ &\Longrightarrow |g(-c)| \cdot |g(x+c) - g(c)| < \epsilon_0 \end{aligned}$$

Then since we must have either $|g(-c)| = 0$ or $|g(-c)| > 0$, consequently one of the following cases applies:

(i) If $|g(-c)| = 0$ then by part (a) we have also $g(y) = 0$ and $g(c) = 0$ so that $|g(y) - g(c)| = 0 < \epsilon$ no matter what δ we choose.

(ii) Otherwise, if $|g(-c)| > 0$ then for any $\epsilon > 0$ we may let $\epsilon_0 = \epsilon|g(-c)|$ and $y = x+c$ so that $\delta = \delta_0 > 0$ exists such that

$$\begin{aligned} |y - c| < \delta &\Longrightarrow |x| < \delta \\ &\Longrightarrow |g(-c)| \cdot |g(x+c) - g(c)| < \epsilon|g(-c)| \\ &\Longrightarrow |g(y) - g(c)| < \epsilon \end{aligned}$$

Hence, in either case, for any $\epsilon > 0$, $\exists \delta$ such that $|y - c| < \delta \Longrightarrow |g(y) - g(c)| < \epsilon$. Therefore g is continuous at c and hence, since c is arbitrary in $\mathbf{R}$, g is continuous at every point of $\mathbf{R}$. **Q.E.D.**