

Real Analysis: Exam II Extra Credit

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3. Problem. Let (x_n) be the sequence defined recursively by $x_1 = 2$ and $x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n}$ for all $n \geq 1$. Show that (x_n) converges and find the limit of the sequence.

To show that (x_n) converges, we will show that (x_n) is bounded below and that (x_n) is ultimately decreasing. It will then follow that (x_n) is bounded above as well and hence, by the Monotone Convergence Theorem, converges.

To show that (x_n) is bounded below, we will show by Mathematical Induction that $x_n > 0$ for all $n \in \mathbf{N}$. We have that $x_1 = 2 > 0$. Suppose $x_k > 0$ for some $k \in \mathbf{N}$. Then $x_k/2 > 0$ and $5/x_k > 0$ so that $x_{k+1} = \frac{x_k}{2} + \frac{5}{x_k} > 0$. Therefore $x_k > 0 \implies x_{k+1} > 0$. Therefore by Mathematical Induction, $x_n > 0$ for all $n \in \mathbf{N}$. Therefore (x_n) is bounded below. Furthermore we can easily see that $x_n \in \mathbf{R}$ for all $n \in \mathbf{N}$.

Now we will show that $x_{n+1} \geq x_{n+2}$ for all $n \in \mathbf{N}$ and hence that the one-tail of (x_n) is decreasing. Let $n \in \mathbf{N}$. Then

$$\begin{aligned}x_{n+1} = \frac{x_n}{2} + \frac{5}{x_n} &\implies x_{n+1} = \frac{x_n^2 + 10}{2x_n} \\ &\implies x_n^2 - 2x_n x_{n+1} + 10 = 0 \\ &\implies x_n^2 - 2x_n x_{n+1} = -10 \\ &\implies x_n^2 - 2x_n x_{n+1} + x_{n+1}^2 = -10 + x_{n+1}^2 \\ &\implies (x_n - x_{n+1})^2 = x_{n+1}^2 - 10 \\ &\implies x_{n+1}^2 - 10 \geq 0 \\ &\implies \frac{x_{n+1}^2}{2} \geq 5 \\ &\implies \frac{x_{n+1}}{2} \geq \frac{5}{x_{n+1}} \text{ since } \forall n \in \mathbf{N}, x_n > 0 \\ &\implies x_{n+1} \geq \frac{x_{n+1}}{2} + \frac{5}{x_{n+1}} \\ &\implies x_{n+1} \geq x_{n+2}\end{aligned}$$

Therefore $x_{n+1} \geq x_{n+2}$ for all $n \in \mathbf{N}$. Therefore $x(n)$ is ultimately decreasing. Therefore $0 < x_n < x_2$ for all $n \in \mathbf{N}$ so that (x_n) is bounded as well as monotone. Therefore by the Monotone Convergence Theorem, (x_n) is convergent.

To find $\lim(x_n)$, let $x = \lim(x_n) = \lim(x_{n+1})$ by virtue of the fact that any tail of (x_n) also converges to x . Then,

$$\begin{aligned}x &= \lim(x_{n+1}) = \frac{\lim(x_n)}{2} + \frac{5}{\lim(x_n)} \\ \implies x &= \frac{x}{2} + \frac{5}{x} \\ \implies 2x^2 &= x^2 + 10 \\ \implies x^2 &= 10 \\ \implies x &= \sqrt{10} \text{ since } \forall n \in \mathbf{N}, x_n > 0\end{aligned}$$

Therefore $\lim(x_n) = \sqrt{10}$.

Q.E.D.