Real Analysis: Chapter 3.6

David Joseph Stith

In the exercises that follow, we will use the following previously established results:

2.4.5 Corollary. *If $t > 0$, there exists $n_t \in \mathbf{N}$ such that $0 < 1/n_t < t$.*

3.6.4 Theorem. *Let (x_n) and (y_n) be two sequences of real numbers and suppose that $x_n \leq y_n$ for all $n \in \mathbf{N}$.*

(a) *If $\lim(x_n) = +\infty$, then $\lim(y_n) = +\infty$.*

(b) *If $\lim(y_n) = -\infty$, then $\lim(x_n) = -\infty$.*

Page 88, Number 1. Exercise. *Give examples of properly divergent sequences (x_n) and (y_n) with $y \neq 0$ for all $n \in \mathbf{N}$ such that*

(a) *(x_n/y_n) is convergent.*

Let $x_n := n$ and $y_n := n$. Then we know (x_n) and (y_n) are properly divergent with $\lim(x_n) = \lim(y_n) = +\infty$, but $x_n/y_n = n/n = 1$ so that (x_n/y_n) is a constant sequence which therefore converges.

(b) *(x_n/y_n) is properly divergent.*

Let $x_n := n^2$ and $y_n = n$. Then we know (y_n) is properly divergent, and (x_n) is shown to be properly divergent by Theorem 3.6.4 since $x_n \geq y_n$ for all $n \in \mathbf{N}$. Then $x_n/y_n = n^2/n = n = y_n$ so that (x_n/y_n) is properly divergent as well.

Page 88, Number 3. Exercise. *Show that if $x_n > 0$ for all $n \in \mathbf{N}$, then $\lim(x_n) = 0$ if and only if $\lim(1/x_n) = +\infty$.*

Suppose that $x_n > 0$ for all $n \in \mathbf{N}$. We need to show both of the following:

(i) $\lim(x_n) = 0 \Longrightarrow \lim(1/x_n) = +\infty$.

(ii) $\lim(1/x_n) = +\infty \Longrightarrow \lim(x_n) = 0$.

We proceed as follows:

(i) Suppose $\lim(x_n) = 0$. We need to show that for every $\alpha \in \mathbf{R}$, $\exists K(\alpha)$ such that $n \geq K(\alpha) \Longrightarrow 1/x_n > \alpha$. But for any $\alpha <= 0$ we have $x_n > 0 \Longrightarrow 1/x_n > 0 >= \alpha$ for all n. Hence we need only to find $K(\alpha)$ for every $\alpha > 0$. Suppose $\alpha > 0$ and let $\epsilon = 1/\alpha$. Then $\exists K_0(\epsilon)$ such that $|x_n| < \epsilon$ so that

$$\begin{aligned} |x_n| < \epsilon &\Longrightarrow x_n < \epsilon \\ &\Longrightarrow \frac{1}{\epsilon} < \frac{1}{x_n} \quad \text{since } x_n > 0 \\ &\Longrightarrow \alpha < \frac{1}{x_n} \end{aligned}$$

Therefore for every $\alpha > 0$ we can choose $K(\alpha) = K_0(1/\alpha)$ so that $n \geq K(\alpha) \Longrightarrow 1/x_n > \alpha$. Therefore $\lim(1/x_n) = +\infty$.

(ii) Suppose $\lim(1/x_n) = +\infty$. We need to show that for every $\epsilon > 0$, there exists a $K_0(\epsilon)$ such that $\forall n \geq K_0(\epsilon)$, $|x_n| < \epsilon$ for all $n \geq K_0(\epsilon)$. We know that for every $\alpha > 0$, $\exists K(\alpha)$ such that $n \geq K(\alpha) \implies 1/x_n > \alpha$. Then since $x_n > 0$ we have

$$\frac{1}{x_n} > \alpha \implies \frac{1}{\alpha} > x_n \implies |x_n| < \frac{1}{\alpha}$$

Therefore by Corollary 2.4.5, for every $\epsilon > 0$ we can choose an α such that $\frac{1}{\alpha} < \epsilon$ and then let $K_0(\epsilon) = K(\alpha)$ so that for all $n \geq K_0(\epsilon)$,

$$|x_n| < \frac{1}{\alpha} < \epsilon.$$

Hence $\lim(x_n) = 0$

Q.E.D.

Page 88, Number 5. Exercise. *Is the sequence* $(n \sin n)$ *properly divergent?*

No, the sequence $(n \sin n)$ is not properly divergent. Let $x_n = n \sin n$. For every $k \in \mathbf{N}$, $\exists n \geq K$ such that $\sin n > 0$ and thus $x_n > 0$ and $\exists m \geq K$ such that $\sin m < 0$ and thus $x_m < 0$. Therefore for any $\alpha > 0$ there does not exist a $K(\alpha) \in \mathbf{N}$ such that $n \geq K(\alpha) \implies x_n > \alpha$ and likewise for any $\beta < 0$ there does not exist a $K(\beta) \in \mathbf{N}$ such that $n \geq K(\beta) \implies x_n < \beta$.

Therefore (x_n) is not properly divergent. **Q.E.D.**