# Real Analysis: Chapter 3.5

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In the exercises that follow, we will use the following previously established results:

**3.2.3a Theorem.** *Let $X = (x_n)$ and $Y = (y_n)$ be sequences of real numbers that converge to $x$ and $y$, respectively, and let $x \in \mathbf{R}$. Then the sequences $X + Y$, $X - Y$, $X \cdot Y$, and $cX$ converge to $x + y$, $x - y$, $xy$, and $cx$, respectively.*

**3.2.3b Theorem.** *If $X = (x_n)$ converges to $x$ and $Z = (z_n)$ is a sequence of nonzero real numbers that converges to $z$ and if $z \neq 0$, then the quotient sequence $X/Z$ converges to $x/z$.*

**3.5.4 Lemma.** *A Cauchy sequence of real numbers is bounded.*

**Page 86, Number 1. Exercise.** *Give an example of a bounded sequence that is not a Cauchy sequence.*

We will show that $(x_n)$ defined by $x_n := (-1)^n$ is bounded but is not a Cauchy sequence.

Since $x_n \in \{1, -1\}$ for all $n \in \mathbf{N}$, $(x_n)$ is bounded.

Since $|x_n - x_{n+1}| = |(-1)^n - (-1)^{n+1}| = |(-1)^n(1+1)| = 2$ for all $n \in \mathbf{N}$, consequently there does not exist a natural number $H(\epsilon)$ when $0 < \epsilon < 2$ such that for all natural numbers $n, m \geq H(\epsilon)$, $|x_n - x_m| < \epsilon$. Therefore $(x_n)$ is not a Cauchy sequence. **Q.E.D.**

**Page 86, Number 2. Exercise.** *Show directly from the definition that the following are Cauchy sequences.*

**(a)** $\left(\dfrac{n+1}{n}\right)$

Let $x_n = \frac{n+1}{n}$ and suppose $\epsilon > 0$. We need to show that there exists a natural number $H(\epsilon)$ such that for all natural numbers $n, m \geq H(\epsilon)$, $|x_n - x_m| < \epsilon$.

We have

$$\begin{aligned} |x_n - x_m| &= \left|\frac{n+1}{n} - \frac{m+1}{m}\right| \\ &= \left|1 + \frac{1}{n} - 1 - \frac{1}{m}\right| \\ &= \left|\frac{1}{n} - \frac{1}{m}\right| \\ &\leq \left|\frac{1}{n}\right| + \left|\frac{1}{m}\right| \end{aligned}$$

so that choosing $H(\epsilon) > \frac{2}{\epsilon}$, we have

$$n > \frac{2}{\epsilon} \Longrightarrow \epsilon > \frac{2}{n} \Longrightarrow \frac{\epsilon}{2} > \frac{1}{n}$$

and likewise for $m$. Therefore

$$|x_n - x_m| \leq \frac{1}{n} + \frac{1}{m} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $(x_n)$ is a Cauchy sequence. **Q.E.D.**

**(b)** $\left(1 + \frac{1}{2!} + \cdots + \frac{1}{n!}\right)$

Let $x_n = 1 + \frac{1}{2!} + \cdots + \frac{1}{n!}$ and suppose $\epsilon > 0$. We need to show that there exists a natural number $H(\epsilon)$ such that for all natural numbers $n, m \geq H(\epsilon)$, $|x_n - x_m| < \epsilon$.

If $m = n$ then $|x_n - x_m| = 0 < \epsilon$. Otherwise, without loss of generality, we may assume that $n > m$ so that

$$|x_n - x_m| = \left|\frac{1}{(m+1)!} + \frac{1}{(m+2)!} + \cdots + \frac{1}{n!}\right|$$

Then, when $m \geq 3$ so that $\frac{1}{k!} < \frac{1}{2^k}$ for all natural numbers $k > m$,

$$\begin{aligned}
|x_n - x_m| &\leq \left|\frac{1}{2^{m+1}} + \frac{1}{2^{m+2}} + \cdots + \frac{1}{2^n}\right| \\
&= \left|\frac{1}{2^{m+1}}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^{n-m}}\right)\right| \\
&= \left|\frac{1}{2^{m+1}}\left(\frac{1 - (1/2)^{n-m+1}}{1/2}\right)\right| \\
&\leq \left|\frac{2}{2^{m+1}}\right| \\
&= \frac{1}{2^m}
\end{aligned}$$

so that choosing $H(\epsilon) > \frac{\ln 1/\epsilon}{\ln 2}$, we have

$$\begin{aligned}
m > \frac{\ln 1/\epsilon}{\ln 2} &\Longrightarrow m \ln 2 > \ln \frac{1}{\epsilon} \\
&\Longrightarrow e^{m \ln 2} > \frac{1}{\epsilon} \\
&\Longrightarrow 2^m > \frac{1}{\epsilon} \\
&\Longrightarrow \epsilon > \frac{1}{2^m}
\end{aligned}$$

Therefore $|x_n - x_m| < \epsilon$.

Therefore $(x_n)$ is a Cauchy sequence. **Q.E.D.**

**Page 86, Number 3. Exercise.** *Show directly from the definition that the following are not Cauchy sequences.*

**(a)** $((-1)^n)$

Let $x_n = (-1)^n$ and let $\epsilon = 1$. We need to show that $(x_n)$ is not a Cauchy sequence. To do this, we will show that for any $H \in \mathbf{N}$ there exists natural numbers $m, n \geq H$ such that $|x_n - x_m| > \epsilon$.

Take $H$ to be any natural number and let $m = H$ and $n = m + 1$. Then

$$|x_n - x_m| = |(-1)^{m+1} - (-1)^m| = |(-1)^m(-1-1)| = 2 > \epsilon$$

Therefore for any $H \in \mathbf{N}$ there exists natural numbers $m, n \geq H$ such that $|x_n - x_m| > \epsilon$.

Therefore $(x_n)$ is not a Cauchy sequence. **Q.E.D.**

**(b)** $\left(n + \dfrac{(-1)^n}{n}\right)$

Let $x_n = n + \dfrac{(-1)^n}{n}$ and let $\epsilon = 1$. We need to show $(x_n)$ is not a Cauchy sequence. To do this, we will show that for any $H \in \mathbf{N}$ there exists natural numbers $m, n \geq H$ such that $|x_n - x_m| > \epsilon$.

Take $H$ to be any natural number and let $m > H$ such that $m$ is even. Then let $n = m + 4$ so that $n$ is also even. Then

$$\begin{aligned}
|x_n - x_m| &= \left|n + \frac{1}{n} - m - \frac{1}{m}\right| \\
&= \left|m + 4 + \frac{1}{n} - m - \frac{1}{m}\right| \\
&= \left|4 + \frac{1}{n} - \frac{1}{m}\right| \\
&\geq \left|4 - \frac{1}{m}\right| \\
&\geq |4 - 1| \quad \text{since } m \in N \\
&> \epsilon
\end{aligned}$$

Therefore for any $H \in \mathbf{N}$ there exists natural numbers $m, n \geq H$ such that $|x_n - x_m| > \epsilon$.

Therefore $(x_n)$ is not a Cauchy sequence. **Q.E.D.**

**(c)** $(\ln n)$

Let $x_n = \ln n$. We need to show that $(x_n)$ is not a Cauchy sequence. To do this we will show that for any $\epsilon > 0$ and any $H \in \mathbf{N}$, there exists an $m, n \geq H(\epsilon)$ such that $|x_n - x_m| > \epsilon$.

Suppose $\epsilon > 0$. Then for any $H \in \mathbf{N}$, we can choose $m \geq H$ and $n > e^{\epsilon + \ln m}$ so that

$$\begin{aligned} |x_n - x_m| &> \left|\ln e^{\epsilon + \ln m} - \ln m\right| \\ &= |\epsilon + \ln m - \ln m| \\ &= |\epsilon| = \epsilon \end{aligned}$$

Therefore for any $H \in \mathbf{N}$ there exist natural numbers $m, n \geq H$ such that $|x_n - x_m| > \epsilon$.

Therefore $(x_n)$ is not a Cauchy sequence. **Q.E.D.**

**Page 86, Number 4. Exercise.** *Show directly from the definition that if* $(x_n)$ *and* $(y_n)$ *are Cauchy sequences, then* $(x_n + y_n)$ *and* $(x_n y_n)$ *are Cauchy sequences.*

Suppose $(x_n)$ and $(y_n)$ are Cauchy sequences. Then for every $\epsilon_x > 0$ there exists an $H_x(\epsilon_x) \in \mathbf{N}$ such that $\forall m, n \geq H_x(\epsilon_x)$, $|x_n - x_m| < \epsilon_x$ and for every $\epsilon_y > 0$ there exists an $H_y(\epsilon_y)$ such that $\forall m, n \geq H_y(\epsilon_y)$, $|y_n - y_m| < \epsilon_y$. We need to show that for every $\epsilon > 0$ there exists an $H(\epsilon)$ such that $\forall m, n \geq H(\epsilon)$, $|x_n + y_n - x_m - y_m| < \epsilon$.

To do so, we choose $H(\epsilon) = \sup\{H_x(\epsilon/2), H_y(\epsilon/2)\}$ so that $n, m \geq H(\epsilon)$ implies that

$$\begin{aligned} |x_n + y_n - x_m - y_m| &= |(x_n - x_m) + (y_n - y_m)| \\ &\leq |x_n - x_m| + |y_n - y_m| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon \end{aligned}$$

Therefore $(x_n + y_n)$ is a Cauchy sequence.

It remains to show that $(x_n y_n)$ is a Cauchy sequence. To do this we will show that for every $\epsilon > 0$ there exists an $H(\epsilon)$ such that $\forall m, n \geq H(\epsilon)$, $|x_n y_n - x_m y_m| < \epsilon$. Suppose $\epsilon > 0$.

We have,

$$\begin{aligned} |x_n y_n - x_m y_m| &= |x_n y_n - x_n y_m + x_n y_m - x_m y_m| \\ &= |x_n(y_n - y_m) + y_m(x_n - x_m)| \\ &\leq |x_n(y_n - y_m)| + |y_m(x_n - x_m)| \end{aligned}$$

Now, since $(x_n)$ and $(y_n)$ are Cauchy sequences, and hence by Lemma 3.5.4 are bounded, we know there exists a $K_x$ such that for all $n, m > K_x$,

$$|x_n - x_m| < \frac{\epsilon}{2\sup\{y_n\}}$$

and likewise there exists a $K_y$ such that for all $n, m > K_y$,

$$|y_n - y_m| < \frac{\epsilon}{2\sup\{x_n\}}$$

Therefore, for all $n, m > \sup\{K_x, K_y\}$,

$$\begin{aligned}|x_n y_n - x_m y_m| &< \frac{|x_n|\epsilon}{2\sup\{x_n\}} + \frac{|y_n|\epsilon}{2\sup\{y_n\}} \\ &\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\ &= \epsilon\end{aligned}$$

Therefore $(x_n y_n)$ is a Cauchy sequence.

**Q.E.D.**

**Page 86, Number 5. Exercise.** *If* $x_n = \sqrt{n}$*, show that* $(x_n)$ *satisfies* $\lim |x_{n+1} - x_n| = 0$*, but that it is not a Cauchy sequence.*

Suppose $x_n = \sqrt{n}$. First we will show that $\lim |x_{n+1} - x_n| = 0$.
We have

$$\begin{aligned}x_{n+1} - x_n &= \sqrt{n+1} - \sqrt{n} \\ &= \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1/\sqrt{n}}{\sqrt{\frac{n+1}{n}} + 1} \\ &= \frac{\sqrt{\frac{1}{n}}}{\sqrt{1 + \frac{1}{n}} + 1}\end{aligned}$$

so that since $\lim(1/n) = 0$ we have

$$\lim |x_{n+1} - x_n| = \frac{\sqrt{\lim(1/n)}}{\sqrt{1 + \lim 1/n} + 1} = \frac{\sqrt{0}}{\sqrt{1+0} + 1} = 0$$

It remains to show that $(x_n)$ is not a Cauchy sequence. To do this we will show by contradiction that $(x_n)$ is not bounded. Then it will follow from the contrapositive of Lemma 3.5.4 that $(x_n)$ is not a Cauchy sequence.

Suppose that $(x_n)$ is bounded above by $M$. Then by the Archimedean property we can choose $n > M^2$. Then $n > M^2 \implies \sqrt{n} > M \implies x_n > M$. But this contradicts our supposition that $M$ is an upper bound of $(x_n)$. Thus $(x_n)$ cannot be bounded. Therefore $(x_n)$ is not a Cauchy sequence. **Q.E.D.**

**Page 86, Number 7. Exercise.** *Let $(x_n)$ be a Cauchy sequence such that $x_n$ is an integer for every $n \in \mathbf{N}$. Show that $(x_n)$ is ultimately constant.*

To show that $(x_n)$ is ultimately constant, we will show that there exists a $K \in \mathbf{N}$ such that for all natural numbers $n, m \geq K$, $x_n = x_m$.

Let $\epsilon = 1/2$. Then since $(x_n)$ is a Cauchy sequence, we know that $\exists H(\epsilon)$ such that $\forall n, m \geq H(\epsilon)$,

$$\begin{aligned} |x_n - x_m| < \epsilon = \frac{1}{2} &\Longrightarrow -\frac{1}{2} < x_n - x_m < \frac{1}{2} \\ &\Longrightarrow x_m - \frac{1}{2} < x_n < x_m + \frac{1}{2} \\ &\Longrightarrow x_n = x_m \quad \text{since } x_n, x_m \in \mathbf{Z} \end{aligned}$$

Therefore we can choose $K = H(\epsilon)$ so that $\forall n, m \geq K$, $x_n = x_m$.

Therefore $(x_n)$ is ultimately constant. **Q.E.D.**

**Page 86, Number 9. Exercise.** *If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbf{N}$, show that $(x_n)$ is a Cauchy sequence.*

To show that $(x_n)$ is a Cauchy sequence, we will show that for all $\epsilon > 0$, $\exists H(\epsilon) \in \mathbf{N}$ such that $\forall n, m \geq H(\epsilon)$, $|x_n - x_m| < \epsilon$.

The trivial case when $n = m$ gives $|x_n - x_m| = 0 < \epsilon$. Otherwise without loss of generality we may assume $n > m$ and proceed as follows.

$$\begin{aligned} |x_n - x_m| &= |x_n - x_{n-1} + x_{n-1} - x_{n-2} + x_{n-2} - \cdots + x_{m+1} - x_m| \\ &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m)| \\ &= |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\ &\leq r^{n-1} + r^{n-2} + \cdots + r^m \\ &= r^m(1 + r + r^2 + \cdots + r^{n-m-1}) \\ &= r^m \left( \frac{1 - r^{n-m}}{1 - r} \right) \\ &\leq r^m \left( \frac{1}{1 - r} \right) \quad \text{since } 0 < r < 1 \Longrightarrow 0 < r^{n+m} < 1 \end{aligned}$$

Therefore by choosing $H(\epsilon) > \frac{\ln(1-r)\epsilon}{\ln r}$ we have

$$\begin{aligned} m > \frac{\ln(1-r)\epsilon}{\ln r} &\Longrightarrow m \ln r < \ln(1-r)\epsilon \quad \text{since } \ln r < 0 \\ &\Longrightarrow r^m < (1-r)\epsilon \\ &\Longrightarrow \frac{r^m}{1-r} < \epsilon \end{aligned}$$

Hence $|x_n - x_m| < \epsilon$ for all $n, m \geq H(\epsilon)$.

Therefore $(x_n)$ is a Cauchy sequence. **Q.E.D.**

**Page 86, Number 10. Exercise.** *If* $x_1 < x_2$ *are arbitrary real numbers and* $x_n = \frac{1}{2}(x_{n-2} + x_{n-1})$ *for* $n > 2$*, show that* $(x_n)$ *is convergent. What is its limit?*

First we will show that

$$x_n = \frac{2x_2 + x_1}{3} + \frac{4(-1/2)^n(x_2 - x_1)}{3} \tag{10.1}$$

by Mathematical Induction.

Let $P(n)$ be the statement (10.1) above. $P(1)$ is true since

$$\frac{2x_2 + x_1}{3} + \frac{4(-1/2)^1(x_2 - x_1)}{3} = \frac{2x_2 + x_1 - 2x_2 + 2x_1}{3} = x_1$$

Now suppose $P(n)$ is true for $n \in \{1, 2, \ldots, k\}$. Then

$$\begin{aligned} x_k + 1 &= \frac{1}{2}(x_{k-1} + x_k) \\ &= \frac{1}{2}\left[2\left(\frac{2x_2 + x_1}{3}\right) + \frac{4(-1/2)^{k-1}(x_2 - x_1)}{3}\left(1 + \left(-\frac{1}{2}\right)\right)\right] \\ &= \frac{2x_2 + x_1}{3} - \frac{1}{2}\left(\frac{4(-1/2)^k(x_2 - x_1)}{3}\right) \\ &= \frac{2x_2 + x_1}{3} - \frac{4(-1/2)^{k+1}(x_2 - x_1)}{3} \end{aligned}$$

so that $P(k+1)$ is true.

Therefore $P(n)$ is true for all $N \in \mathbf{N}$.

Now we will show that $\lim(x_n) = \frac{2x_2 + x_1}{3}$. To do this, we will show that

$$\lim\left((-1/2)^n\right) = 0$$

and hence that

$$\lim(x_n) = \frac{2x_2 + x_1}{3} + \frac{4 \cdot 0 \cdot (x_2 - x_1)}{3} = \frac{2x_2 + x_1}{3}$$

by Theorem 3.2.3.

To show that $\lim\left((-1/2)^n\right) = 0$ we will show that for every $\epsilon > 0$, there exists a $K(\epsilon) \in \mathbf{N}$ such that for all natural numbers $n \geq K(\epsilon)$, $|(-1/2)^n| < \epsilon$.

We have $|(-1/2)^n| = \frac{1}{2^n} < \frac{1}{n}$ so that by choosing $K(\epsilon) > \frac{1}{\epsilon}$ we have

$$n > \frac{1}{\epsilon} \Longrightarrow \epsilon > \frac{1}{n} \Longrightarrow \left|\left(-\frac{1}{2}\right)^n\right| < \epsilon$$

Therefore $\lim\left((-1/2)^n\right) = 0$. Therefore $\lim(x_n) = \frac{2x_2 + x_1}{3}$.

Therefore $(x_n)$ is convergent. **Q.E.D.**