Real Analysis: Chapter 3.4

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In the exercises that follow, we will use the following previously established results:

3.3.2 Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*
(a) If $X = (x_n)$ is a bounded increasing sequence, then

$$\lim(x_n) = \sup\{x_n : n \in \mathbf{N}\}$$

(b) If $Y = (y_n)$ is a bounded decreasing sequence, then

$$\lim(y_n) = \inf\{y_n : n \in \mathbf{N}\}$$

3.4.2 Theorem. *If a sequence $X = (x_n)$ of real numbers converges to a real number x, then any subsequence $X' = (x_{n_k})$ of X also converges to x.*

Page 80, Number 1. Exercise. *Give an example of an unbounded sequence that has a convergent subsequence.*

Let

$$x_n := \frac{(-1)^n n + n}{2}$$

Then (x_n) is unbounded since for any $M \in \mathbf{R}$, there exists an even $n > M$ with $x_n = 2n/2 = n > M$.

However, $x_k = 0$ for all odd natural numbers k so that the subsequence (x_{2n-1}) obviously converges. **Q.E.D.**

Page 80, Number 2. Exercise. *Use the method of Example 3.4.3(b) to show that if $0 < c < 1$, then $\lim(c^{\frac{1}{n}}) = 1$.*

We have $0 < c < 1 \Longrightarrow 0 < c^{\frac{1}{n}} < 1$ so that $(c^{\frac{1}{n}})$ is bounded. Furthermore

$$\begin{aligned} c < 1 &\Longrightarrow \frac{1}{c} > 1 \\ &\Longrightarrow \left(\frac{1}{c}\right)^{\frac{1}{n+1}} < \left(\frac{1}{c}\right)^{\frac{1}{n}} \text{ since } \frac{1}{n+1} < \frac{1}{n} \\ &\Longrightarrow \frac{1}{c^{\frac{1}{n+1}}} < \frac{1}{c^{n}} \\ &\Longrightarrow c^{n} < c^{\frac{1}{n+1}} \end{aligned}$$

so that $(c^{\frac{1}{n}})$ is increasing. Therefore by the Monotone Convergence Theorem, $(c^{\frac{1}{n}})$ is convergent. Then by Theorem 3.4.2, it follows that $\lim(c^{\frac{1}{n}}) = \lim(c^{\frac{1}{2n}})$. Then, since

$$c^{\frac{1}{2n}} = \left(c^{\frac{1}{n}}\right)^{\frac{1}{2}} \Longrightarrow c^{\frac{1}{n}} = \left(c^{\frac{1}{2n}}\right)^2$$

consequently

$$\lim(c^{\frac{1}{n}}) = \left(\lim(c^{\frac{1}{n}})\right)^2$$

Letting $x = \lim(c^{\frac{1}{n}})$, we then have $x = x^2 \Longrightarrow x = 0 \vee x = 1$. But (x_n) is increasing, therefore $x_1 = c < x_n$ for all $n \in \mathbf{N}$. Therefore $x = 1$. Therefore $\lim(x_n) = 1$. **Q.E.D.**

Page 80, Number 3. Exercise. *Let (f_n) be the Fibonacci sequence*

$$f_1 := 1, f_2 := 1, f_{n+1} := f_{n-1} + f_n \text{ for } n \geq 2$$

and let $x_n := f_{n+1}/f_n$. Given that $\lim(x_n) = L$ exists, determine the value of L.

We have,

$$\begin{aligned} x_{n+1} &= \frac{f_{n+2}}{f_{n+1}} \\ &= \frac{f_n + f_{n+1}}{f_{n+1}} \\ &= \frac{f_n}{f_{n+1}} + 1 \\ &= \frac{1}{f_{n+1}/f_n} + 1 \\ &= \frac{1}{x_n} + 1 \end{aligned}$$

Therefore

$$\lim(x_{n+1}) = \frac{1}{\lim(x_n)} + 1 = \lim(x_n)$$

so that letting $x = \lim(x_n)$,

$$\begin{aligned} x = \frac{1}{x} + 1 &\Longrightarrow x^2 = 1 + x \\ &\Longrightarrow x^2 - x - 1 = 0 \\ &\Longrightarrow x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2} \end{aligned}$$

But since $x_n > 0$ for all $n \in \mathbf{N}$, it must be that $x = (1 + \sqrt{5})/2$. Therefore

$$L = \lim(x_n) = \frac{1 + \sqrt{5}}{2}$$

Q.E.D.

Page 80, Number 5. Exercise. *Let $X = (x_n)$ and $Y = (y_n)$ be given sequences, and let the "shuffled" sequence $Z = (z_n)$ be defined by $z_1 := x_1, z_2 := y_1, \ldots, z_{2n-1} := x_n, z_{2n} := y_n, \ldots$. Show that Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$.*

We must show both that

(i) If Z is convergent then both X and Y are convergent and $\lim X = \lim Y$.

(ii) if X and Y are convergent and $\lim X = \lim Y$ then Z is convergent.

We proceed as follows:

(i) Suppose Z is convergent. We need to show that both X and Y are convergent and $\lim X = \lim Y$. Let $u = \lim Z$. By Theorem 3.4.2 X and Y both converge to u since they are both subsequences of Z. Therefore $\lim X = \lim Y$.

(ii) Suppose X and Y are convergent and $\lim X = \lim Y$. We need to show that Z is convergent. To do this we will show that for every $\epsilon > 0$, there exists a $K(\epsilon)$ such that

$$|z_n - u| < \epsilon \text{ for all } n \geq K(\epsilon)$$

where $u = \lim X = \lim Y$. We know that there exists a $K_x(\epsilon)$ such that

$$|x_n - u| < \epsilon \text{ for all } n \geq K_x(\epsilon)$$

and a $K_y(\epsilon)$ such that

$$|y_n - u| < \epsilon \text{ for all } n \geq K_y(\epsilon)$$

since $\lim X = \lim Y = u$. Therefore by choosing $K(\epsilon) > 2\max\{K_x(\epsilon), K_y(\epsilon)\}$ we have that

$$n > K(\epsilon) \Longrightarrow z_n \in \{x_k : k \geq K_x(\epsilon)\} \cup \{y_k : k \geq K_y(\epsilon)\}$$

so that $|z_n - u| < \epsilon$. Therefore $\lim(z_n) = u$ and hence Z is convergent.

Therefore Z is convergent if and only if both X and Y are convergent and $\lim X = \lim Y$. **Q.E.D.**

Page 80, Number 6. Exercise. *Let $x_n := n^{\frac{1}{n}}$ for $n \in \mathbf{N}$.*

(a) *Show that $x_{n+1} < x_n$ if and only if $(1 + 1/n)^n < n$, and infer that the inequality is valid for $n \geq 3$. Conclude that (x_n) is ultimately decreasing and that $x = \lim(x_n)$ exists.*

First we need to show that

(i) $x_{n+1} < x_n \Longrightarrow (1 + 1/n)^n < n$

(ii) $(1 + 1/n)^n \Longrightarrow x_{n+1} < x_n$

Proceeding thus:

(i) Suppose $x_{n+1} < x_n$. Then

$$\begin{aligned} x_{n+1} < x_n &\implies (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n}} \\ &\implies (n+1)^{\frac{n}{n+1}} < n \\ &\implies \left[n\left(1+\frac{1}{n}\right)\right]^{\frac{n}{n+1}} < n \\ &\implies n^{\frac{n}{n+1}}\left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} < n \\ &\implies \left(1+\frac{1}{n}\right)^{\frac{n}{n+1}} < \frac{n}{n^{\frac{n}{n+1}}} = n^{1-\frac{n}{n+1}} = n^{\frac{1}{n+1}} \\ &\implies \left(1+\frac{1}{n}\right)^{n} < n \end{aligned}$$

Therefore $x_{n+1} < x_n \implies (1+1/n)^n < n$

(ii) Now suppose that $(1+1/n)^n < n$. We need to show that $x_{n+1} < x_n$. We have,

$$\begin{aligned} \left(1+\frac{1}{n}\right)^{n} < n &\implies \left(1+\frac{1}{n}\right)^{\frac{1}{n+1}} < n^{\frac{1}{n(n+1)}} \\ &\implies \left(\frac{n+1}{n}\right)^{\frac{1}{n+1}} < n^{\frac{1}{n(n+1)}} \\ &\implies (n+1)^{\frac{1}{n+1}} < n^{\frac{1}{n+1}+\frac{1}{n(n+1)}} \\ &\implies (n+1)^{\frac{1}{n+1}} < n^{\frac{n+1}{n(n+1)}} = n^{\frac{1}{n}} \\ &\implies x_{n+1} < x_n \end{aligned}$$

Therefore $(1+1/n)^n \implies x_{n+1} < x_n$.

Now we need to show that $(1+1/n)^n < 3$ for $n \geq 3$ and hence that $x_{n+1} < x_n$ for $n \geq 3$. To do this we will use Mathematical Induction. Let $P(n)$ be the statement $(1+1/n)^n < n$. Then $P(3)$ means

$$(1+\frac{1}{3})^3 < 3 \iff \frac{4^3}{3^3} < 3 \iff \frac{48}{27} < 3$$

hence $P(3)$ is true. Now suppose $P(k)$ is true for some $k \geq 3$. Then $(1+1/k)^k < k$. We need to show that $(1+\frac{1}{k+1})^{k+1} < k+1$. We have,

$$\begin{aligned} \left(1+\frac{1}{k}\right)^{k} &\implies \left(1+\frac{1}{k}\right)^{k}\left(1+\frac{1}{k}\right) < k\left(1+\frac{1}{k}\right) \\ &\implies \left(1+\frac{1}{k}\right)^{k+1} < k+1 \\ &\implies \left(1+\frac{1}{k+1}\right)^{k+1} < k+1 \\ &\implies P(k+1) \end{aligned}$$

Therefore $P(n)$ is true for all $n \geq 3, n \in \mathbf{N}$. Therefore $(1+1/n)^n < n$ and hence $x_{n+1} < x_n$ for all $n \geq 3, n \in \mathbf{N}$. Therefore (x_n) is ultimately decreasing. Therefore it follows from the Monotone Convergence Theorem that (x_n) is convergent and hence that $x = \lim(x_n)$ exists. **Q.E.D.**

(b) *Use the fact that the subsequence (x_{2n}) also converges to x to conclude that $x = 1$.*

We have,

$$x_{2n} = (2n)^{1/2n} = \left[(2n)^{1/n}\right]^{1/2} = \left[2^{1/n}n^{1/n}\right]^{1/2} = \left[2^{1/n}x_n\right]^{1/2}$$

and we know that $\lim(2^{1/n}) = 1$ and that $x = \lim(x_n) = \lim(x_{2n})$ exists. Therefore $x = 1 \cdot x^{1/2}$ so that $x = 0$ or $x = 1$. But since $x_n = n^{1/n} > 1$ for all $n \in \mathbf{N}$, consequently $x = 1$. **Q.E.D.**