Real Analysis: Chapter 3.3

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In the exercises that follow, we will use the following previously established results:

2.3.4 Lemma. *An upper bound* u *of a nonempty set* S *in* $\mathbf{R}$ *is the supremum of* S *if and only if for every* $\epsilon > 0$ *there exists an* $s_\epsilon \in S$ *such that* $u - \epsilon < s_\epsilon$*.*

3.2.7 Squeeze Theorem. *Suppose that* $X = (x_n)$, $Y = (y_n)$, *and* $Z = (z_n)$ *are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \quad \textit{for all } n \in \mathbf{N},$$

and that $\lim(x_n) = \lim(z_n)$*. Then* $Y = (y_n)$ *is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

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3.3.2 Monotone Convergence Theorem. *A monotone sequence of real numbers is convergent if and only if it is bounded. Further:*
(a) If $X = (x_n)$ *is a bounded increasing sequence, then*

$$\lim(x_n) = \sup\{x_n : n \in \mathbf{N}\}$$

(b) If $Y = (y_n)$ *is a bounded decreasing sequence, then*

$$\lim(y_n) = \inf\{y_n : n \in \mathbf{N}\}$$

Page 74, Number 1. Exercise. *Let* $x_1 := 8$ *and* $x_{n+1} := \frac{1}{2}x_n + 2$ *for* $n \in \mathbf{N}$. *Show that* (x_n) *is bounded and monotone. Find the limit.*

We will show that $x_n > 0$ for all $n \in \mathbf{N}$, and that $x_{n+1} < x_n$ for all $n \in \mathbf{N}$.

Since $x_1 = 8 > 0$ and for any $x_k > 0$, $x_{k+1} = \frac{1}{2}x_k + 2 > 0$, consequently $x_n > 0$ for all $n \in \mathbf{N}$. Therefore (x_n) is bounded below by 0.

It remains to show that (x_n) is monotone and bounded above. To do this, we will show that (x_n) is decreasing and hence that $0 < x_n < x_1$. We will show that (x_n) is decreasing by Mathematical Induction. Let $P(n)$ be the statement $x_n \geq x_{n+1}$. Since $x_2 = \frac{1}{2}(8)+2 = 6$, $x_1 \geq x_2$ so that $P(1)$ is clearly true. Now suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then $x_k \geq x_{k+1}$. We need to show that $x_{k+1} \geq x_{k+2}$ and hence that $P(k+1)$ is true. We have

$$\begin{aligned} x_k \geq x_{k+1} &\Longrightarrow \frac{1}{2}x_k \geq \frac{1}{2}x_{k+1} \\ &\Longrightarrow \frac{1}{2}x_k + 2 \geq \frac{1}{2}x_{k+1} + 2 \\ &\Longrightarrow x_{k+1} \geq x_{k+2} \end{aligned}$$

Therefore $P(k) \Longrightarrow P(k+1)$. Therefore $P(n)$ is true for all $n \in \mathbf{N}$. Therefore (x_n) is decreasing. Therefore (x_n) is monotone and bounded.

Therefore by the Monotone Convergence Theorem (x_n) is convergent so that $\lim(x_n)$ exists. Then the sequence (x_{n+1}) has the same limit since (x_{n+1}) is a tail of (x_n). Then

$$\lim(x_{n+1}) = \frac{1}{2}\lim(x_n) + 2 = \lim(x_n)$$

so that $\lim(x_n) = 4$. **Q.E.D.**

Page 74, Number 2. Exercise. *Let* $x_1 > 1$ *and* $x_{n+1} = 2 - \frac{1}{x_n}$ *for* $n \in \mathbf{N}$. *Show that* (x_n) *is bounded and monotone. Find the limit.*

We will show that $1 < x_n$ for all $n \in \mathbf{N}$. To do this we will use Mathematical Induction.

Let $P(n)$ be the statement $1 < x_n$. $P(2)$ is true since $x_1 > 1 \Longrightarrow 1 < 2 - \frac{1}{x_1}$. Now suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then $1 < x_k$. We need to show that $1 < x_{k+1}$. Now since $1 < x_k$, $1/x_k < 1$ so that $2 - 1/x_n > 1$. Therefore $1 < x_{k+1}$ so that (x_n) is bounded below.

To show that (x_n) is monotone, we will show that (x_n) is decreasing. It will then follow that (x_n) is bounded above as well.

We know that $x_n > 1$ for all $n \in \mathbf{N}$. Then,

$$\begin{aligned} x_n > 1 &\Longrightarrow x_n - 1 > 0 \\ &\Longrightarrow (x_n - 1)^2 > 0 \\ &\Longrightarrow x_n^2 - 2x_n + 1 > 0 \\ &\Longrightarrow x_n^2 - x_n > x_n - 1 \\ &\Longrightarrow x_n - 1 > 1 - \frac{1}{x_n} \\ &\Longrightarrow x_n > 2 - \frac{1}{x_n} \\ &\Longrightarrow x_n > x_{n+1} \end{aligned}$$

Therefore (x_n) is decreasing. Therefore $x_n \leq x_1$ so that (x_n) is bounded with $1 < x_n \leq x_1$. Therefore by the Monotone Convergence Theorem (x_n) converges so that $\lim(x_n)$ exists. Then the sequence (x_{n+1}) has the same limit since (x_{n+1}) is a tail of (x_n). Then

$$\lim(x_{n+1}) = 2 - \frac{1}{\lim(x_n)} = \lim(x_n)$$

so that letting $x = \lim(x_n)$ we have

$$\begin{aligned} x = 2 - \frac{1}{x} &\Longrightarrow x^2 = 2x - 1 \\ &\Longrightarrow x^2 - 2x + 1 = 0 \\ &\Longrightarrow (x - 1)^2 = 0 \\ &\Longrightarrow x = 1 \end{aligned}$$

Therefore $\lim(x_n) = 1$. **Q.E.D.**

Page 74, Number 4. Exercise. *Let* $x_1 := 1$ *and* $x_{n+1} := \sqrt{2 + x_n}$ *for* $n \in \mathbf{N}$. *Show that* (x_n) *converges and find the limit.*

We will show that $x_n < 2$ for all $n \in \mathbf{N}$ and that (x_n) is increasing. Then it will follow that $x_1 \leq x_n < 2$, that (x_n) is bounded, and hence by the Monotone Convergence Theorem that (x_n) converges.

To show that $x_1 < 2$ we will use Mathematical Induction. Let $P(n)$ be the statement $x_n < 2$. $P(1)$ is obviously true since $x_1 = 1 < 2$. Now suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned} x_n < 2 &\Longrightarrow 2 + x_n < 4 \\ &\Longrightarrow \sqrt{2 + x_n} < 2 \\ &\Longrightarrow x_{n+1} < 2 \\ &\Longrightarrow P(k + 1) \end{aligned}$$

Therefore $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $x_n < 2$ for all $n \in \mathbf{N}$.

It remains to show that (x_n) is increasing. To do this we will use Mathematical Induction. Let $Q(n)$ be the statement $x_n \le x_{n+1}$. $Q(1)$ is true since $1 \le \sqrt{2+1}$. Now suppose $Q(k)$ is true for some $k \in \mathbf{N}$. Then $x_k \le x_{k+1}$. We need to show that $x_{k+1} \le x_{k+2}$. We have

$$\begin{aligned} x_k \le x_{k+1} &\Longrightarrow 2 + x_k \le 2 + x_{k+1} \\ &\Longrightarrow \sqrt{2+x_k} \le \sqrt{2+x_{k+1}} \\ &\Longrightarrow x_{k+1} \le x_{k+2} \end{aligned}$$

Therefore $Q(k) \Longrightarrow Q(k+1)$. Therefore $Q(n)$ is true for all $n \in \mathbf{N}$. Therefore $x_n \le x_{n+1}$ for all $n \in \mathbf{N}$. Therefore (x_n) is increasing. Therefore by the Monotone Convergence Theorem, (x_n) is convergent. Therefore $\lim(x_n)$ exists. Then (x_{n+1}) converges to the same limit since (x_{n+1}) is a tail of (x_n). So,

$$\lim(x_{n+1}) = \sqrt{2 + \lim(x_n)} = \lim(x_n)$$

Letting $x = \lim(x_n)$ we have

$$\begin{aligned} x = \sqrt{2+x} &\Longrightarrow x^2 = 2 + x \wedge x \ge 0 \\ &\Longrightarrow x^2 - x - 2 = 0 \wedge x \ge 0 \\ &\Longrightarrow (x+1)(x-2) = 0 \wedge x \ge 0 \\ &\Longrightarrow x = 2 \end{aligned}$$

Therefore $\lim(x_n) = 2$. **Q.E.D.**

Page 74, Number 5. Exercise. *Let* $y_1 := \sqrt{p}$, *where* $p > 0$, *and* $y_{n+1} := \sqrt{p + y_n}$ *for* $n \in \mathbf{N}$. *Show that* (y_n) *converges and find the limit.*

First we will show that $0 \le y_n \le 1 + 2\sqrt{p}$ for all $n \in \mathbf{N}$. To do this we will use Mathematical Induction. Let $P(n)$ be the statement $0 \le y_n \le 1 + 2\sqrt{p}$. We have

$$p > 0 \Longrightarrow \sqrt{p} > 0 \Longrightarrow y_1 > 0$$

and

$$y_1 = \sqrt{p} \Longrightarrow y_1 \le 2\sqrt{p} \Longrightarrow y_1 \le 1 + 2\sqrt{p}$$

so that $P(1)$ is true. Now suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then $0 \le y_k \le 1 + 2\sqrt{p}$. We need to show that $0 \le y_{k+1} \le 1 + 2\sqrt{p}$. Since $y_{k+1} = \sqrt{p + y_k}$ is always non-negative, $0 \le y_{k+1}$. And since

$$y_{k+1} \le \sqrt{p + 1 + 2\sqrt{p}} \le \sqrt{(\sqrt{p}+1)^2} \le 1 + \sqrt{p} \le 1 + 2\sqrt{p}$$

we have $0 \le y_{k+1} \le 1 + 2\sqrt{p}$ so that $P(k+1)$ is true. Therefore $P(k) \Longrightarrow P(k+1)$. Therefore $P(n)$ is true for all $n \in \mathbf{N}$. Therefore (y_n) is bounded.

It remains to show that (y_n) converges. To do this we will show that (y_n) is monotone as well as bounded and hence, by the Monotone Convergence Theorem, is convergent.

To show that (y_n) is monotone we will use Mathematical Induction to show that (y_n) is increasing. Let $Q(n)$ be the statement $y_n \leq y_{n+1}$. Then

$$y_2 = \sqrt{p + y_1} = \sqrt{p + \sqrt{p}} \geq \sqrt{p} \geq y_1$$

so that $P(1)$ is true. Now suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then

$$\begin{aligned} P(k) &\implies y_k \leq y_{k+1} \\ &\implies p + y_k \leq p + y_{k+1} \\ &\implies \sqrt{p + y_k} \leq \sqrt{p + y_{k+1}} \\ &\implies y_{k+1} \leq y_{k+2} \\ &\implies P(k+1) \end{aligned}$$

so that $P(k) \implies P(k+1)$. Therefore $P(n)$ is true for all $n \in \mathbf{N}$. Therefore (y_n) is increasing. Therefore since (y_n) is bounded and monotone, (y_n) is convergent. Therefore $\lim(y_n)$ exists. Then (y_{n+1}) converges to the same limit since (y_{n+1}) is a tail of (y_n). So

$$\lim(y_{n+1}) = \sqrt{p + \lim(y_n)} = \lim(y_n)$$

Letting $y = \lim(y_n)$ we have

$$\begin{aligned} y = \sqrt{p + y} &\implies y^2 = p + y \wedge y \geq 0 \\ &\implies y^2 - y - p = 0 \wedge y \geq 0 \\ &\implies y = \frac{1 \pm \sqrt{1 + 4p}}{2} \wedge y \geq 0 \\ &\implies y = \frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p} \end{aligned}$$

Therefore (y_n) converges to $\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p}$. **Q.E.D.**

Page 74, Number 7. Exercise. *Let $x_1 := a > 0$ and $x_{n+1} := x_n + 1/x_n$ for $n \in \mathbf{N}$. Determine if (x_n) converges or diverges.*

We will show that (x_n) diverges. To do this we will show that the convergence of (x_n) is absurd. Suppose (x_n) converges. Then $\lim(x_n)$ exists. Let $x = \lim(x_n)$. Then $x = \lim(x_{n+1})$ since (x_{n+1}) is a tail of (x_n). Then

$$x = x + \frac{1}{x} \implies 0 = \frac{1}{x} \implies 0 = 1$$

This is absurd. Hence it cannot be true that (x_n) converges. Therefore (x_n) diverges. **Q.E.D.**

Page 74, Number 9. Exercise. *Let A be an infinite subset of $\mathbf{R}$ that is bounded above and let $u := \sup A$. Show there exists an increasing sequence (x_n) with $x_n \in A$ for all $n \in \mathbf{N}$ such that $u = \lim(x_n)$.*

Let us consider the two cases $u \in A$ and $u \notin A$ separately.

(i) If $u \in A$, then define (x_n) to be the constant sequence $x_n := u$. Then (x_n) is increasing since $x_{n+1} \geq x_n$ for all $n \in \mathbf{N}$ and $x_n \in A$ since $u \in A$. Furthermore, $\lim(x_n) = u$ so that all conditions given for (x_n) have been met.

(ii) If $u \notin A$, then construct (x_n) as follows: We know by Lemma 2.3.4 that for every $\epsilon > 0$ there exists an $a_\epsilon \in A$ such that $u - \epsilon < a_\epsilon$. Let $x_1 > u - k$ for any $k > 0$. Then choose each $x_n \in A$ such that $\max\{x_{n-1}, u - \frac{1}{n}\} < x_n < u$. Now x_n exists for all $n \in \mathbf{N}$ by Lemma 2.3.4, and (x_n) is clearly increasing. It remains only to show that $\lim(x_n) = u$. To do this, we will show that for any $\epsilon > 0$, there exists a $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|x_n - u| < \epsilon$$

By choosing $K(\epsilon) > \frac{1}{\epsilon}$, we have $n > \frac{1}{\epsilon} \Longrightarrow \epsilon > \frac{1}{n}$. But

$$u - \frac{1}{n} < x_n < u \Longrightarrow -\frac{1}{n} < x_n - u < \frac{1}{n} \Longrightarrow |x_n - u| < \frac{1}{n}$$

Therefore $|x_n - u| < \frac{1}{n} < \epsilon$ so that $\lim(x_n) = u$. **Q.E.D.**

Page 74, Number 10. Exercise. *Let (x_n) be a bounded sequence, and for each $n \in \mathbf{N}$ let $s_n := \sup\{x_k : k \geq n\}$ and $t_n := \inf\{x_k : k \geq n\}$. Prove that (s_n) and (t_n) are monotone and convergent. Also prove that if $\lim(s_n) = \lim(t_n)$, then (x_n) is convergent.*

First we will show that (s_n) is monotone. To do this we will show that $s_n \geq s_{n+1}$ for all $n \in \mathbf{N}$. Consider s_n for some $n \in \mathbf{N}$. We have $s_n = \sup\{x_k : k \geq n\}$. Then $s_{n+1} = \sup\{x_k : k \geq n+1\}$. Hence $s_{n+1} = \sup\{\{x_k : k \geq n\} \setminus \{x_n\}\}$. Therefore if $s_n \neq x_n$ then $s_{n+1} = s_n$ but if $s_n = x_n$ then $s_{n+1} < s_n$. Therefore $s_n \geq s_{n+1}$ for all $n \in \mathbf{N}$.

To see that (s_n) is convergent, simply note that (s_n) is also bounded since (x_n) is bounded, then it follows from the Monotone Convergence Theorem that (x_n) is convergent.

Likewise, we show that (t_n) is monotone by showing that $t_n \leq t_{n+1}$ for all $n \in \mathbf{N}$. Consider t_n for some $n \in \mathbf{N}$. We have $t_n = \inf\{x_k : k \geq n\}$. Then $t_{n+1} = \inf\{x_k : k \geq n + 1\}$. Hence $t_{n+1} = \inf\{\{x_k : k \geq n\} \setminus \{x_n\}\}$. Therefore if $t_n \neq x_n$ then $t_{n+1} = t_n$ but if $t_n = x_n$ then $t_{n+1} > t_n$. Therefore $t_n \leq t_{n+1}$ for all $n \in \mathbf{N}$.

And likewise (t_n) is also bounded since (x_n) is bounded so that (t_n) is convergent.

It remains to show that if $\lim(s_n) = \lim(t_n)$ then (x_n) is convergent. To do this, we will use the Squeeze Theorem. We have $t_n \leq x_n \leq s_n$ for all $n \in \mathbf{N}$ since t_n is a lower bound and s_n is an upper bound of a set that contains x_n. Therefore since $\lim(t_n) = \lim(s_n)$, by the Squeeze Theorem we know that $\lim(t_n) = \lim(x_n) = \lim(s_n)$ so that (x_n) is convergent. **Q.E.D.**