Real Analysis: Chapter 3.2

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In the exercises that follow, we will use the following previously established results:

3.1.6d Example. *If* $0 < b < 1$, *then* $\lim (b^n) = 0$

3.1.11d Example. $\lim \left(n^{\frac{1}{n}} \right) = 1$

3.2.3a Theorem. *Let* $X = (x_n)$ *and* $Y = (y_n)$ *be sequences of real numbers that converge to* x *and* y*, respectively, and let* $x \in \mathbf{R}$*. Then the sequences* $X + Y$, $X - Y$, $X \cdot Y$, *and* cX *converge to* $x + y$, $x - y$, xy, *and* cx*, respectively.*

3.2.3b Theorem. *If* $X = (x_n)$ *converges to* x *and* $Z = (z_n)$ *is a sequence of nonzero real numbers that converges to* z *and if* $z \neq 0$*, then the quotient sequence* X/Z *converges to* x/z*.*

3.2.7 Squeeze Theorem. *Suppose that* $X = (x_n)$, $Y = (y_n)$, *and* $Z = (z_n)$ *are sequences of real numbers such that*

$$x_n \leq y_n \leq z_n \quad \text{for all } n \in \mathbf{N},$$

and that $\lim(x_n) = \lim(z_n)$*. Then* $Y = (y_n)$ *is convergent and*

$$\lim(x_n) = \lim(y_n) = \lim(z_n)$$

.

3.2.9 Theorem. *Let the sequence* $X = (x_n)$ *converge to* x*. Then the sequence* $(|x_n|)$ *of absolute values converges to* $|x|$*.*

3.2.10 Theorem. *Let* $X = (x_n)$ *be a sequence of real numbers that converges to* x *and suppose that* $x_n \geq 0$*. Then the sequence* $\left(\sqrt{x_n}\right)$ *of positive square roots converges and* $\lim \left(\sqrt{x_n}\right) = \sqrt{x}$*.*

Page 67, Number 1. Exercise. *For* x_n *given by the following formulas, establish either the convergence or the divergence of the sequence* $X = (x_n)$*.*

(a) $x_n := \dfrac{n}{n+1}$

We will show that $\lim(x_n) = 1$ using Theorems 3.2.3a and 3.2.3b.
We have,

$$x_n = \frac{n}{n+1} = \frac{1}{1 + \frac{1}{n}}$$

And we know $\lim(1) = 1$ and $\lim\left(\frac{1}{n}\right) = 0$.
Therefore

$$\lim(x_n) = \lim\left(\frac{1}{1+\frac{1}{n}}\right) = \frac{\lim(1)}{\lim(1) + \lim\left(\frac{1}{n}\right)} = \frac{1}{1+0} = 1$$

Q.E.D.

(b) $x_n := \dfrac{(-1)^n n}{n+1}$

We will show that (x_n) diverges. To do this we will show that $\lim(x_n)$ does not exist. Suppose to the contrary that $\lim(x_n)$ does exist. Then let $a = \lim(x_n)$ and let $\epsilon = \frac{1}{2}$. Then by our supposition that $\lim(x_n)$ exists, there would exist a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|x_n - a| < \epsilon$$

Then,

$$\begin{aligned} |x_n - a| < \epsilon &\Longrightarrow |a - x_n| < \epsilon \\ &\Longrightarrow \left| a - \frac{(-1)^n}{n+1} \right| < \frac{1}{2} \\ &\Longrightarrow -\frac{1}{2} < a - \frac{(-1)^2}{n+1} < \frac{1}{2} \\ &\Longrightarrow -\frac{1}{2} + \frac{(-1)^2}{n+1} < a < \frac{1}{2} + \frac{(-1)^2}{n+1} \end{aligned}$$

so that for some even $x > n$, $-\frac{1}{2} + \frac{x}{x+1} < a$ and for some odd $y > n$, $a < \frac{1}{2} - \frac{y}{y+1}$. But

$$x > 1 \Longrightarrow 2x > x + 1 \Longrightarrow \frac{x}{x+1} > \frac{1}{2} \Longrightarrow -\frac{1}{2} + \frac{x}{x+1} > 0 \Longrightarrow a > 0$$

And

$$y > 1 \Longrightarrow 2y > y + 1 \Longrightarrow \frac{y}{y+1} > \frac{1}{2} \Longrightarrow 0 > \frac{1}{2} - \frac{y}{y+1} \Longrightarrow a < 0$$

Hence by the Trichotomy property a cannot exist. Therefore $\lim(x_n)$ does not exist. Therefore (x_n) diverges.

(c) $x_n := \dfrac{n^2}{n+1}$

We will show that (x_n) diverges. To do this, we will show that (x_n) is not bounded. Then, we know for any sequence of real numbers that if the sequence converges then the sequence is bounded. Hence the contrapositive of this statement is true, namely that if the sequence is not bounded then the sequence diverges.

Now, for any $n \in \mathbf{N}$ we have

$$\begin{aligned} 0 > -1 &\Longrightarrow n^2 > n^2 - 1 \\ &\Longrightarrow n^2 > (n+1)(n-1) \\ &\Longrightarrow \frac{n^2}{n+1} > n - 1 \\ &\Longrightarrow x_n > n - 1 \end{aligned}$$

Therefore for any proposed bound M for (x_n), we know there exists a natural number $n > M + 1$ so that $M < n - 1 < x_n$ and hence M cannot be a bound for (x_n). Therefore x_n is not bounded. Therefore (x_n) diverges.

(d) $x_n := \dfrac{2n^2+3}{n^2+1}$

We will show that $\lim(x_n) = 2$ using Theorems 3.2.3a and 3.2.3b.
We have,

$$\begin{aligned} x_n = \frac{2n^2+3}{n^2+1} &= \frac{2+\frac{3}{n^2}}{1+\frac{1}{n^2}} \\ &= \frac{2+3(\frac{1}{n})(\frac{1}{n})}{1+(\frac{1}{n})(\frac{1}{n})} \end{aligned}$$

And we know that $\lim(c) = c$ for any constant c, and $\lim(1/n) = 0$.

Therefore by recursive application of Theorems 3.2.3a and 3.2.3b we have

$$\begin{aligned} \lim(x_n) &= \frac{\lim(2) + 3\lim\left(\frac{1}{n}\right)\lim\left(\frac{1}{n}\right)}{\lim(1) + \lim\left(\frac{1}{n}\right)\lim\left(\frac{1}{n}\right)} \\ &= \frac{2+3(0)(0)}{1+(0)(0)} = 2 \end{aligned}$$

Q.E.D.

Page 67, Number 2. Exercise. *Give an example of two divergent sequences X and Y such that*

(a) *their sum $X + Y$ converges.*

Let $X = (1, -1, 1, -1, \ldots)$.
Let $Y = (-1, 1, -1, 1, \ldots)$.
Then it is easy to see that X and Y both diverge, yet

$$X + Y = (0, 0, 0, 0, \ldots)$$

converges to 0.

(b) *their product XY converges.*

The X and Y given above both diverge, yet

$$XY = (-1, -1, -1, -1, \ldots)$$

converges to -1.

Page 67, Number 3. Exercise. *Show that if X and Y are sequences such that X and $X + Y$ are convergent, then Y is convergent.*

Suppose $X = (x_n)$ and $Y = (y_n)$ are sequences and that X and $X+Y$ are convergent. Then by Theorem 3.2.3 we know that $(X + Y) - X$ is convergent. Then,

$$\begin{aligned} (X+Y) - X &= ((x_n + y_n) - x_n) \\ &= (y_n) \\ &= Y \end{aligned}$$

Hence Y is convergent. **Q.E.D.**

Page 67, Number 5. Exercise. *Show that the following sequences are not convergent.*

(a) (2^n)

To show that (2^n) is not convergent, we will show that it is not bounded. Then, we know for any sequence of real numbers that if the sequence converges then the sequence is bounded. Hence the contrapositive of this statement is true, namely that if the sequence is not bounded then the sequence is not convergent.

To show that (2^n) is not bounded, we will use mathematical induction to show that for any $m \in \mathbf{N}$ there exists an $n \in \mathbf{N}$ such that $2^n > m$. Clearly, when $m = 1$, $2^1 > m$. Now suppose that for some $k \in \mathbf{N}$, there exists an n_k such that $2^{n_k} > k$. Then we consider whether there exists an n_{k+1} such that $2^{n_{k+1}} > k + 1$. We know that $2^{n_k} \geq k + 1$ since $2^{n_k} > k$. If $2^{n_k} > k + 1$ then simply let $n_{k+1} = n_k$. Otherwise, $2^{n_k} = k + 1$ so that $n_{k+1} = n_k + 1$ suffices to make $2^{n_{k+1}} = 2(k + 1) > k + 1$. Hence the existence of n_k such that $2^{n_k} > k$ implies the existence of n_{k+1} such that $2^{n_{k+1}} > k + 1$. Therefore by mathematical induction, for any $m \in \mathbf{N}$ there exists an $n \in \mathbf{N}$ such that $2^n > m$.

Therefore (2^n) is not bounded. Therefore (2^n) is not convergent.

(b) $\big((-1)^n n^2\big)$

To show that $\big((-1)^n n^2\big)$ is not convergent, we will show that it is not bounded. Then, as we have seen earlier, it follows that the sequence cannot converge.

We know that the set of natural numbers is not bounded. Therefore, since $2n > n$ for all $n \in \mathbf{N}$, the set of even natural numbers is not bounded. Then, since $n^2 \geq n$ and $(-1)^n = 1$ for all n in the set of even natural numbers, consequently $(-1)^n n^2 > n$ so that $\{((-1)^n n^2) : n \text{ is even}\}$ is not bounded. Therefore, since every element of this set is contained in the sequence $\big((-1)^n n^2\big)$, consequently this sequence is not bounded.

Therefore $\big((-1)^n n^2\big)$ is not convergent. **Q.E.D.**

Page 67, Number 7. Exercise. *If* (b_n) *is a bounded sequence and* $\lim(a_n) = 0$, *show that* $\lim(a_n b_n) = 0$. *Explain why Theorem 3.2.3* **cannot** *be used.*

Suppose (b_n) is a bounded sequence and (a_n) is a convergent sequence with $\lim(a_n) = 0$. We need to show that $\lim(a_n b_n) = 0$. To do this we will show that for any $\epsilon > 0$, there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|a_n b_n| < \epsilon$$

Since (b_n) is bounded, we know there exists an M such that $|b_n| \leq M$ for all $n \in \mathbf{N}$. Then,

$$\begin{aligned} |b_n| \leq M &\Longrightarrow |b_n| \leq |M| \quad \text{since } M \leq |M| \\ &\Longrightarrow |a_n||b_n| \leq |a_n||M| \\ &\Longrightarrow |a_n b_n| \leq |a_n M| \end{aligned}$$

And since $\lim(a_n) = 0$, we know there exists a $K(\epsilon/|M|)$ such that for all $n \geq K(\epsilon/|M|)$,

$$|a_n| < \frac{\epsilon}{|M|}$$

Then, $|a_n||M| < \epsilon$ and therefore $|a_n M| < \epsilon$. Therefore $|a_n b_n| \leq |a_n M| < \epsilon$. Therefore $\lim(a_n b_n) = 0$. **Q.E.D.**

We could not use Theorem 3.2.3 because we do not know that (b_n) is convergent.

Page 67, Number 9. Exercise. *Let* $y_n := \sqrt{n+1} - \sqrt{n}$ *for* $n \in \mathbf{N}$. *Show that* (y_n) *and* $(\sqrt{n} y_n)$ *converge. Find their limits.*

First we will show that $\lim(y_n) = 0$ and hence that (y_n) converges.

We have,

$$\begin{aligned} y_n &= \sqrt{n+1} - \sqrt{n} \\ &= \frac{\left(\sqrt{n+1} - \sqrt{n}\right)\left(\sqrt{n+1} + \sqrt{n}\right)}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{(n+1) - n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1/\sqrt{n}}{\sqrt{n+1}/\sqrt{n} + 1} \\ &= \frac{\sqrt{1/n}}{\sqrt{1 + 1/n} + 1} \end{aligned}$$

We know that $\lim(1) = 1$ and $\lim(1/n) = 0$. Therefore by recursive application of Theorems 3.2.3a, 3.2.3b, and 3.2.10,

$$\begin{aligned} \lim(y_n) &= \frac{\sqrt{0}}{\sqrt{1+0} + 1} \\ &= 0 \end{aligned}$$

Therefore (y_n) converges to 0.

Next we will show that $\lim\left(\sqrt{n}y_n\right) = 1/2$ and hence that $\sqrt{n}y_n$ converges.
We have,

$$\begin{aligned}
\sqrt{n}y_n &= \sqrt{n}\sqrt{n+1} - n \\
&= \frac{\left(\sqrt{n^2+n} - n\right)\left(\sqrt{n^2+n} + n\right)}{\sqrt{n^2+n} + n} \\
&= \frac{n^2 + n - n^2}{\sqrt{n^2+n} + n} \\
&= \frac{n}{\sqrt{n^2+n} + n} \\
&= \frac{1}{\frac{\sqrt{n^2+n}}{n} + 1} \\
&= \frac{1}{\sqrt{\frac{n+1}{n}} + 1} \\
&= \frac{1}{\sqrt{1 + \frac{1}{n}} + 1}
\end{aligned}$$

We know that $\lim(1) = 1$ and $\lim(1/n) = 0$. Therefore by recursive application of Theorems 3.2.3a, 3.2.3b, and 3.2.10,

$$\begin{aligned}
\lim\left(\sqrt{n}y_n\right) &= \frac{1}{\sqrt{1+0}+1} \\
&= \frac{1}{2}
\end{aligned}$$

Therefore $(\sqrt{n}y_n)$ converges to $\frac{1}{2}$. **Q.E.D.**

Page 67, Number 10. Exercise. *Determine the following limits:*

(a) $\lim\left((3\sqrt{n})^{\frac{1}{2n}}\right)$

We will show that $\lim\left((3\sqrt{n})^{\frac{1}{2n}}\right) = 1$. To do this we will show that $\lim\left(3^{\frac{1}{2n}}\right) = 1$ and that $\lim\left(n^{\frac{1}{4n}}\right) = 1$. Then, since

$$(3\sqrt{n})^{\frac{1}{2n}} = 3^{\frac{1}{2n}}(n^{\frac{1}{2}})^{\frac{1}{2n}} = 3^{\frac{1}{2n}} \cdot n^{\frac{1}{4n}}$$

it will follow from Theorem 3.2.3a that

$$\lim\left((3\sqrt{n})^{\frac{1}{2n}}\right) = \lim\left(3^{\frac{1}{2n}}\right)\lim\left(n^{\frac{1}{4n}}\right) = 1 \cdot 1 = 1$$

To show that $\lim\left(3^{\frac{1}{2n}}\right) = 1$, we will show that for every $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n > K(\epsilon)$,

$$\left|3^{\frac{1}{2n}} - 1\right| < \epsilon$$

By the Archimedean property, there exists $K(\epsilon) > \log_{\epsilon+1}\sqrt{3}$ for which

$$\begin{aligned} n > \log_{\epsilon+1}\sqrt{3} &\Longrightarrow (\epsilon+1)^n > \sqrt{3} \quad \text{since } \epsilon+1>1 \\ &\Longrightarrow [(\epsilon+1)^n]^{\frac{1}{n}} > \left[3^{\frac{1}{2}}\right]^{\frac{1}{n}} \quad \text{since } \tfrac{1}{n}>0 \\ &\Longrightarrow \epsilon+1 > 3^{\frac{1}{2n}} \\ &\Longrightarrow \epsilon > 3^{\frac{1}{2n}} - 1 \\ &\Longrightarrow \epsilon > \left|3^{\frac{1}{2n}} - 1\right| \quad \text{since } \tfrac{1}{2n} > 0 \Longrightarrow 3^{\frac{1}{2n}} > 0 \end{aligned}$$

Therefore $\left|3^{\frac{1}{2n}} - 1\right| < \epsilon$ for all $n \geq K(\epsilon)$ and hence $\lim(3^{\frac{1}{2n}}) = 1$.

It remains to show that $\lim(n^{\frac{1}{4n}}) = 1$. To do this, we will use the Squeeze Theorem. First note that since $n > 0$,

$$0 \leq n \leq 4n \Longrightarrow 0 \leq \frac{1}{4n} \leq \frac{1}{n} \Longrightarrow n^0 \leq n^{\frac{1}{4n}} \leq n^{\frac{1}{n}}$$

We know that $\lim(n^0) = \lim(1) = 1$ and we know that $\lim(n^{\frac{1}{n}}) = 1$ by Example 3.1.11d. Hence it follows from the Squeeze Theorem that $\lim(n^{\frac{1}{4n}}) = 1$. (Note that a proof that $\lim\left(3^{\frac{1}{2n}}\right) = 1$ using the Squeeze theorem is also possible since for $n \geq 3$, $3^0 \leq 3^{\frac{1}{2n}} \leq n^{\frac{1}{2n}} \leq n^{\frac{1}{n}}$.)

Therefore $\lim\left(\left(3\sqrt{n}\right)^{\frac{1}{2n}}\right) = 1$. **Q.E.D.**

(b) $\lim\left((n+1)^{\frac{1}{\ln(n+1)}}\right)$

We will show that $\lim\left((n+1)^{\frac{1}{\ln(n+1)}}\right) = e$.

Let $x = (n+1)^{\frac{1}{\ln(n+1)}}$.
Then

$$\begin{aligned} x = (n+1)^{\frac{1}{\ln(n+1)}} &\Longrightarrow x^{\ln(n+1)} = n+1 \\ &\Longrightarrow \ln(n+1)\ln(x) = \ln(n+1) \\ &\Longrightarrow \ln(x) = 1 \\ &\Longrightarrow x = e \end{aligned}$$

Therefore $\lim\left((n+1)^{\frac{1}{\ln(n+1)}}\right) = \lim(e) = e$ **Q.E.D.**

Page 67, Number 11. Exercise. *If* $0 < a < b$, *determine* $\lim\left(\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}\right)$.

We will show that $\lim\left(\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) = b$.

We have,

$$\begin{aligned}\frac{a^{n+1}+b^{n+1}}{a^n+b^n} &= \frac{a^{n+1}}{a^n+b^n} + \frac{b^{n+1}}{a^n+b^n}\\ &= \frac{a}{1+b^n/a^n} + \frac{b}{a^n/b^n+1}\\ &= \frac{a\left(\frac{a}{b}\right)^n}{\left(\frac{a}{b}\right)^n+1} + \frac{b}{\left(\frac{a}{b}\right)^n+1}\end{aligned}$$

Then, since $0 < a/b < 1$, we know that $\lim\left((a/b)^n\right) = 0$ by Example 3.1.6d.

Therefore by recursive application of Theorems 3.2.3a and 3.2.3b we have

$$\begin{aligned}\lim\left(\frac{a^{n+1}+b^{n+1}}{a^n+b^n}\right) &= \lim\left(\frac{a\left(\frac{a}{b}\right)^n}{\left(\frac{a}{b}\right)^n+1} + \frac{b}{\left(\frac{a}{b}\right)^n+1}\right)\\ &= \frac{a(0)}{0+1} + \frac{b}{0+1}\\ &= b\end{aligned}$$

Q.E.D.

Page 67, Number 13. Exercise. *Use the Squeeze Theorem 3.2.7 to determine the limits of the following:*

(a) $\left(n^{\frac{1}{n^2}}\right)$

We will show that $\lim\left(n^{\frac{1}{n^2}}\right) = 1$.

First note that since $n > 0$,

$$0 \leq n \leq n^2 \implies 0 \leq \frac{1}{n^2} \leq \frac{1}{n} \implies n^0 \leq n^{\frac{1}{n^2}} \leq n^{\frac{1}{n}}$$

We know that $\lim(n^0) = \lim(1) = 1$ and we know that $\lim(n^{\frac{1}{n}}) = 1$ by Example 3.1.11d. Hence it follows from the Squeeze Theorem that $\lim(n^{\frac{1}{n^2}}) = 1$. **Q.E.D.**

Page 68, Number 21. Exercise. *Suppose that (x_n) is a convergent sequence and (y_n) is such that for any $\epsilon > 0$ there exists M such that $|x_n - y_n| < \epsilon$ for all $n \geq M$. Does it follow that (y_n) is convergent?*

Yes. We will show that it does indeed follow that (y_n) is convergent. To do this, we will show that $\lim(y_n) = \lim(x_n)$.

Let $a = \lim(x_n)$ and let $\epsilon > 0$. We will show that there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|y_n - a| < \epsilon$$

and hence that $\lim(y_n) = a$.

Now, since $\lim(x_n) = a$, we know that for $\frac{1}{2}\epsilon$, M_x exists such that

$$|x_n - a| < \frac{1}{2}\epsilon \quad \text{for all } n \geq M_x.$$

Furthermore, by what is given we know that for $\frac{1}{2}\epsilon$, M_y exists such that

$$|x_n - y_n| < \frac{1}{2}\epsilon \quad \text{for all } n \geq M_y.$$

Therefore $|y_n - x_n| < \frac{1}{2}\epsilon$ for all $n \geq M_y$.

Hence for all $n \geq \sup(M_x, M_y)$,

$$|y_n - x_n| + |x_n - a| < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon$$

Therefore by the Triangle Inequality,

$$|y_n - x_n + x_n - a| \leq |y_n - x_n| + |x_n - a| < \epsilon$$

so that by choosing $K(\epsilon) = \sup(M_x, M_y)$, we have $|y_n - a| < \epsilon$ for all $n \geq K(\epsilon)$ and hence that $\lim(y_n) = a = \lim(x_n)$.

Therefore (y_n) is convergent. **Q.E.D.**

Page 68, Number 22. Exercise. *Show that if (x_n) and (y_n) are convergent sequences, then the sequences (u_n) and (v_n) defined by $u_n := \max\{x_n, y_n\}$ and $v_n := \min\{x_n, y_n\}$ are also convergent.*

Suppose that (x_n) and (y_n) are convergent sequences. Let $u_n := \max\{x_n, y_n\} = \frac{1}{2}(x_n + y_n + |x_n + y_n|)$. Similarly, let $v_n := \min\{x_n, y_n\} = \frac{1}{2}(x_n + y_n - |x_n + y_n|)$. We need to show that (u_n) and (v_n) are convergent.

Now, since (x_n) and (y_n) converge, by Theorem 3.2.3a we know that $(x_n + y_n)$ converges. Therefore by Theorem 3.2.9 we know that $|(x_n + y_n)|$ converges. Therefore by recursive application of Theorem 3.2.3a we know that $\frac{1}{2}(x_n + y_n + |x_n + y_n|)$ converges and that $\frac{1}{2}(x_n + y_n - |x_n + y_n|)$ converges. Therefore (u_n) and (v_n) are convergent. **Q.E.D.**