# Real Analysis: Chapter 3.1

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**Page 59, Number 1. Exercise.** *The sequence* $(x_n)$ *is defined by the following formulas for the* $n^{th}$ *term. Write the first five terms in each case:*

**(a)** $x_n := 1 + (-1)^n$

$$(x_n) = (0, 2, 0, 2, 0, \ldots)$$

**(b)** $x_n := \dfrac{(-1)^n}{n}$

$$(x_n) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\right)$$

**(c)** $x_n := \dfrac{1}{n(n+1)}$

$$(x_n) := \left(\frac{1}{2}, \frac{1}{6}, \frac{1}{12}, \frac{1}{20}, \frac{1}{30}, \ldots\right)$$

**(d)** $x_n := \dfrac{1}{n^2+2}$

$$(x_n) = \left(\frac{1}{3}, \frac{1}{6}, \frac{1}{11}, \frac{1}{18}, \frac{1}{27}, \ldots\right)$$

**Page 59, Number 2. Exercise.** *The first few terms of a sequence* $(x_n)$ *are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the* $n^{th}$ *term* $x_n$*.*

**(a)** $(5, 7, 9, 11, \ldots, 3 + 2n, \ldots)$

**(c)** $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots, \frac{n}{n+1}, \ldots\right)$

**Page 59, Number 3. Exercise.** *List the first five terms of the following inductively defined sequences.*

**(b)** $y_1 := 2, \quad y_{n+1} := \frac{1}{2}(y_n + 2/y_n)$

$$(y_n) = \left(2, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots\right)$$

**(d)** $s_1 := 3, \quad s_2 := 5, \quad s_{n+2} := s_n + s_{n+1}$

$$(s_n) = (3, 5, 8, 13, 21, \ldots)$$

**Page 60, Number 5. Exercise.** *Use the definition of the limit of a sequence to establish the following limits.*

**(b)** $\lim\left(\frac{2n}{n+1}\right) = 2$

Let $(x_n) := \left(\frac{2n}{n+1}\right)$. We need to show that $\lim(x_n) = 2$. To do this, we will show that for every $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|x_n - 2| < \epsilon$$

Suppose $\epsilon > 0$. We have,

$$\begin{aligned} |x_n - 2| &= \left|\frac{2n}{n+1} - 2\right| \\ &= \left|\frac{2n - 2n - 2}{n+1}\right| \\ &= \left|\frac{-2}{n+1}\right| \\ &= \frac{2}{n+1} \\ &< \frac{2}{n} \end{aligned}$$

Hence by choosing $K(\epsilon) > \frac{2}{\epsilon}$ (which we may do by the Archimedean Property), we have

$$n > \frac{2}{\epsilon} \Longrightarrow n\epsilon > 2 \Longrightarrow \epsilon > \frac{2}{n}$$

Therefore, for all $n \geq K(\epsilon)$, $|x_n - 2| < \epsilon$. Therefore $\lim(x_n) = 2$. **Q.E.D.**

**(d)** $\lim\left(\dfrac{n^2-1}{2n^2+3}\right) = \dfrac{1}{2}$

Let $x_n := \left(\frac{n^2-1}{2n^2+3}\right)$. We need to show that $\lim(x_n) = \frac{1}{2}$. To do this, we will show that for every $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$\left|x_n - \frac{1}{2}\right| < \epsilon$$

We have,

$$\begin{aligned}\left|x_n - \frac{1}{2}\right| &= \left|\frac{n^2-1}{2n^2+3} - \frac{1}{2}\right| \\ &= \left|\frac{2n^2-2-2n^2-3}{4n^2+6}\right| \\ &= \left|\frac{-5}{4n^2+6}\right| \\ &= \frac{5}{4n^2+6} \\ &< \frac{5}{4n^2} < \frac{5}{n}\end{aligned}$$

Hence by choosing $K(\epsilon) > \frac{5}{\epsilon}$, we have

$$n > \frac{5}{\epsilon} \Longrightarrow n\epsilon > 5 \Longrightarrow \epsilon > \frac{5}{n}$$

Therefore, for all $n \geq K(\epsilon)$, $\left|x_n - \frac{1}{2}\right| < \epsilon$. Therefore $\lim(x_n) = \frac{1}{2}$. **Q.E.D.**

**Page 60, Number 6. Exercise.** *Show that:*

**(a)** $\lim\left(\dfrac{1}{\sqrt{n+7}}\right) = 0$

Let $(x_n) := \frac{1}{\sqrt{n+7}}$. We need to show that $\lim(x_n) = 0$. To do this, we will show that there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|x_n| < \epsilon$$

Suppose $\epsilon > 0$. We have,

$$\begin{aligned}|x_n| &= \left|\frac{1}{\sqrt{n+7}}\right| \\ &= \frac{1}{\sqrt{n+7}} \\ &< \frac{1}{\sqrt{n}}\end{aligned}$$

Hence by choosing $K(\epsilon) > \frac{1}{\epsilon^2}$, we have

$$n > \frac{1}{\epsilon^2} \Longrightarrow \epsilon^2 > \frac{1}{n} \Longrightarrow \epsilon > \frac{1}{\sqrt{n}}$$

Therefore, for all $n \geq K(\epsilon)$, $|x_n| < \epsilon$. Therefore $\lim(x_n) = 0$. **Q.E.D.**

**(c)** $\lim\left(\frac{\sqrt{n}}{n+1}\right) = 0$

Let $(x_n) := \frac{\sqrt{n}}{n+1}$. We need to show that $\lim(x_n) = 0$. To do this, we will show that there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|x_n| < \epsilon$$

Suppose $\epsilon > 0$. We have,

$$\begin{aligned} |x_n| &= \left|\frac{\sqrt{n}}{n+1}\right| \\ &= \frac{\sqrt{n}}{n+1} \\ &< \frac{\sqrt{n}}{n} \\ &= \frac{1}{\sqrt{n}} \end{aligned}$$

Hence by choosing $K(\epsilon) > \frac{1}{\epsilon^2}$, we have

$$n > \frac{1}{\epsilon^2} \Longrightarrow \epsilon^2 > \frac{1}{n} \Longrightarrow \epsilon > \frac{1}{\sqrt{n}}$$

Therefore, for all $n \geq K(\epsilon)$, $|x_n| < \epsilon$. Therefore $\lim(x_n) = 0$. **Q.E.D.**

**Page 60, Number 8. Exercise.** *Prove that* $\lim(x_n) = 0 \iff \lim(|x_n|) = 0$. *Give an example to show that the convergence of* $(|x_n|)$ *need not imply the convergence of* $(x_n)$.

We need to show both that $\lim(x_n) = 0 \Longrightarrow \lim(|x_n|) = 0$ and that $\lim(|x_n|) \Longrightarrow \lim(x_n) = 0$.

Suppose that $\lim(x_n) = 0$. Then for every $\epsilon > 0$, there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, $|x_n - 0| < \epsilon$. Then,

$$|x_n - 0| = |x_n| = ||x_n|| = ||x_n| - 0|$$

so that $||x_n| - 0| < \epsilon$. Therefore $\lim(|x_n|) = 0$. Therefore $\lim(x_n) = 0 \Longrightarrow \lim(|x_n|) = 0$.

Conversely, suppose that $\lim(|x_n|) = 0$. Then for every $\epsilon > 0$, there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, $||x_n| - 0| < \epsilon$. Then,

$$||x_n| - 0| = ||x_n|| = |x_n| = |x_n - 0|$$

so that $|x_n - 0| < \epsilon$. Therefore $\lim(x_n) = 0$. Therefore $\lim(|x_n|) = 0 \Longrightarrow \lim(x_n) = 0$.

Therefore $\lim(x_n) = 0 \iff \lim(|x_n|) = 0$. **Q.E.D.**

**Page 60, Number 10. Exercise.** *Prove that if* $\lim(x_n) = x$ *and if* $x > 0$*, then there exists a natural number* $M$ *such that* $x_n > 0$ *for all* $n \geq M$*.*

Suppose that $\lim(x_n) = x$ and $x > 0$. Then for every $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$, $|x_n - x| < \epsilon$. Hence

$$\begin{aligned} &-\epsilon < x_n - x < \epsilon \\ \Longrightarrow &-\epsilon + x < x_n < \epsilon + x \end{aligned}$$

Therefore when $\epsilon = x$, which occurs since $x > 0$, then $-\epsilon + \epsilon < x_n$ so that $0 < x_n$. Hence we have that $M = K(x)$ exists such that $x_n > 0$ for all $n \geq M$. **Q.E.D.**

**Page 60, Number 13. Exercise.** *Let* $b \in \mathbf{R}$ *satisfy* $0 < b < 1$*. Show that* $\lim(nb^n) = 0$*.*

Suppose $b \in \mathbf{R}$ such that $0 < b < 1$. We need to show that $\lim(nb^n) = 0$. To do this we will show that for all $\epsilon > 0$ there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$|nb^n| < \epsilon$$

Suppose $\epsilon > 0$. We have $|nb^n| = nb^n$ since $n$ and $b$ are both positive. Now $b$ may be expressed as $\frac{1}{1+a}$ for some $a > 0$ so that

$$nb^n = \frac{n}{(1+a)^n} = \frac{n}{1 + na + \frac{1}{2}n(n-1)a^2 + \cdots}$$

when $n > 1$. Therefore,

$$nb^n \leq \frac{n}{\frac{1}{2}n(n-1)a^2} = \frac{2}{(n-1)a^2}$$

so that by choosing $K(\epsilon) > \frac{2}{a^2\epsilon} + 1$, we have

$$n > \frac{2}{a^2\epsilon} + 1 \Longrightarrow n - 1 > \frac{2}{a^2\epsilon} \Longrightarrow \epsilon > \frac{2}{(n-1)a^2}$$

Hence $|nb^n| < \epsilon$.

Therefore $\lim(nb^n) = 0$. **Q.E.D.**

**Page 60, Number 14. Exercise.** *Show that* $\lim\left((2n)^{\frac{1}{n}}\right) = 1$.

Suppose that $\epsilon > 0$. We need to show that there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$\left|(2n)^{\frac{1}{n}} - 1\right| < \epsilon$$

To do this, we will let $k = (2n)^{\frac{1}{n}} - 1$ and show that $k < \epsilon$. We have,

$$\begin{aligned} k = (2n)^{\frac{1}{n}} - 1 &\Longrightarrow k + 1 = (2n)^{\frac{1}{n}} \\ &\Longrightarrow (k+1)^n = 2n \\ &\Longrightarrow 2n = 1 + nk + \frac{1}{2}n(n-1)k^2 + \cdots \quad \text{for } n > 1 \\ &\Longrightarrow 2n \geq \frac{1}{2}n(n-1)k^2 \\ &\Longrightarrow \frac{4n}{n(n-1)} \geq k^2 \\ &\Longrightarrow \frac{4}{n-1} \geq k^2 \end{aligned}$$

So that by choosing $K(\epsilon) > \frac{4}{\epsilon^2} + 1$, we have

$$\begin{aligned} n > \frac{4}{\epsilon^2} + 1 &\Longrightarrow n - 1 > \frac{4}{\epsilon^2} \\ &\Longrightarrow \epsilon^2 > \frac{4}{n-1} \quad \text{since } n > 1 \\ &\Longrightarrow \epsilon^2 > k^2 \\ &\Longrightarrow \epsilon > k \end{aligned}$$

Therefore $k < \epsilon$. Therefore $\left|(2n)^{\frac{1}{n}} - 1\right| < \epsilon$. Therefore $\lim\left((2n)^{\frac{1}{n}}\right) = 1$. **Q.E.D.**

**Page 60, Number 15. Exercise.** *Show that* $\lim(n^2/n!) = 0$.

Suppose that $\epsilon > 0$. We need to show that there exists a natural number $K(\epsilon)$ such that for all $n \geq K(\epsilon)$,

$$\left|\frac{n^2}{n!}\right| < \epsilon$$

We have,

$$\begin{aligned} \left|\frac{n^2}{n!}\right| &= \frac{n^2}{n!} = \frac{n}{(n-1)!} = \frac{(n-1)+1}{(n-1)!} \\ &= \frac{n-1}{(n-1)!} + \frac{1}{(n-1)!} = \frac{1}{(n-2)!} + \frac{1}{(n-1)!} \quad \text{when } n > 1 \\ &\leq \frac{1}{n-2} + \frac{1}{n-2} = \frac{2}{n-2} \end{aligned}$$

So that by choosing $K(\epsilon) > \frac{2}{\epsilon} + 2$, we have

$$n > \frac{2}{\epsilon} + 2 \Longrightarrow n - 2 > \frac{2}{\epsilon} \Longrightarrow \epsilon > \frac{2}{n-2}$$

Therefore $\left|\frac{n^2}{n!}\right| < \epsilon$. Therefore $\lim\left(\frac{n^2}{n!}\right) = 0$. **Q.E.D.**