Real Analysis: Chapter 2.1

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Page 29, Number 4. Exercise. *If $a \in \mathbf{R}$ satisfies $a \cdot a = a$, prove that either $a = 0$ or $a = 1$.*

Suppose $a \in \mathbf{R}$ and $a \cdot a = a$. Then either $a = 0$ or $a \neq 0$.
Suppose $a = 0$. Then $a \cdot a = a$ is satisfied since we know that $0 \cdot 0 = 0$.
Now suppose $a \neq 0$. Then

$$\begin{aligned} a &= a \cdot 1 \\ &= a \cdot (a \cdot \frac{1}{a}) \\ &= (a \cdot a) \cdot \frac{1}{a} \\ &= a \cdot \frac{1}{a} \quad \text{since } a \cdot a = a \\ &= 1 \end{aligned}$$

Therefore if $a \in \mathbf{R}$ satisfies $a \cdot a = a$, then either $a = 0$ or $a = 1$. **Q.E.D.**

Page 30, Number 6. Exercise. *Use the argument in the proof of Theorem 2.1.4 to show that there does not exist a rational number s such that $s^2 = 6$.*

Suppose p and q are integers such that $(\frac{p}{q})^2 = 6$. Without loss of generality we may assume that p and q are positive and have no common integer factors other than 1. Since $p^2 = 6q^2$, we see that $6 \mid p^2$.

We claim that $6 \mid p^2 \implies 6 \mid p$. To show this we will show that the contrapositive is true:

$$6 \not| p \implies 6 \not| p^2$$

Suppose $6 \not| p$. Then either $2 \not| p$ or $3 \not| p$. We know that $p = a_1^{x_1} a_2^{x_2} \ldots a_k^{x_k}$ for prime factors $a_1 \ldots a_k$ and powers $x_1 \ldots x_k \in \mathbf{N}$ is a unique representation of p. Hence either $a_n \neq 2$ or $a_n \neq 3$ for all $n \in \mathbf{N}$, $n \leq k$. Now, $p^2 = a_1^{2x_1} a_2^{2x_2} \ldots a_k^{2x_k}$ has the same same prime factors as p raised to twice the powers of each. Therefore $6 \not| p \implies 6 \not| p^2$. Therefore $6 \mid p^2 \implies 6 \mid p$.

Now, since p and q have no integer factors other than 1, therefore $6 \not| q$. But we have that $p = 6m$ for some $m \in \mathbf{N}$ and hence $(6m)^2 = 6q^2$ so that $6m^2 = q^2$. Hence $6 \mid q^2$ which implies that $6 \mid q$ thus contradicting our supposition that p and q have no common integer factors other than 1. Therefore it cannot be true that there exists a rational number s such that $s^2 = 6$. **Q.E.D.**

Page 30. Number 14. Exercise. *If* $0 \leq a < b$, *show that* $a^2 \leq ab < b^2$. *Show by example that it does not follow that* $a^2 < ab < b^2$.

Suppose $0 \leq a < b$ for $a, b \in \mathbf{R}$. We need to show that both $a^2 \leq ab$ and $ab < b^2$.

First we will show that $a^2 \leq ab$. We have that either $a = 0$ or $a > 0$. If $a = 0$ then $a^2 = ab$ since $a^2 = 0$ and $ab = 0$. Otherwise, if $a > 0$ then $a < b \Longrightarrow a^2 < ab$. Therefore $a^2 = ab$ or $a^2 < ab$ so that $a^2 \leq ab$.

Now we will show that $ab < b^2$. We have that either $a = 0$ and hence that $ab = 0 < b$ or that $0 < a$ and $a < b$ and hence again that $0 < b$. Then $a < b$ and $0 < b$ together imply that $ab < b^2$.

Therefore $a^2 \leq ab < b^2$. **Q.E.D.**

To see that it does not follow that $a^2 < ab < b^2$ consider the case when $a = 0$ and $b = 1$. Then it is true that $0 \leq a < b$ but we have $a^2 = ab$ since $0^2 = 0 \cdot 1$. Hence it is not true that $a^2 < ab$ and therefore it does not follow that $a^2 < ab < b^2$. **Q.E.D.**

Page 30. Number 16. Exercise. *Find all real numbers* x *that satisfy the following inequalities.*

(a) $x^2 > 3x + 4$

$$\begin{aligned} x^2 > 3x + 4 &\Longrightarrow x^2 - 3x - 4 > 0 \\ &\Longrightarrow (x-4)(x+1) > 0 \\ &\Longrightarrow (x - 4 > 0 \wedge x + 1 > 0) \vee (x - 4 < 0 \wedge x + 1 < 0) \\ &\Longrightarrow (x > 4 \wedge x > -1) \vee (x < 4 \wedge x < -1) \\ &\Longrightarrow \boxed{(x > 4) \vee (x < -1)} \end{aligned}$$

(b) $1 < x^2 < 4$

Since $x^2 = (-x)^2$,

$$\begin{aligned} 1 < x^2 < 4 &\Longrightarrow \begin{cases} 1 < x < 2 & \text{if } x \geq 0 \\ 1 < -x < 2 & \text{if } x < 0 \end{cases} \\ &\Longrightarrow \boxed{(1 < x < 2) \vee (-2 < x < -1)} \end{aligned}$$

(c) $\frac{1}{x} < x$

We need only consider the two cases when $x > 0$ or when $x < 0$ since when $x = 0$, $\frac{1}{x}$ is undefined.

When $x > 0$ we have

$$x > \frac{1}{x} \Longrightarrow x^2 > 1 \Longrightarrow x > 1$$

Otherwise, when $x < 0$ we have

$$x > \frac{1}{x} \Longrightarrow x^2 < 1$$

Then, let $y = -x$ so that $y > 0$ and

$$\begin{aligned} x^2 < 1 &\Longrightarrow (-y)^2 < 1 \\ &\Longrightarrow y^2 < 1 \\ &\Longrightarrow y < 1 \\ &\Longrightarrow -x < 1 \\ &\Longrightarrow x > -1 \end{aligned}$$

Therefore $-1 < x < 0$.

Therefore $\boxed{(-1 < x < 0) \vee (x > 1)}$.

(d) $\frac{1}{x} < x^2$

We need only consider the two cases when $x > 0$ or when $x < 0$ since when $x = 0$, $\frac{1}{x}$ is undefined.

When $x > 0$ we have

$$x^2 > \frac{1}{x} \Longrightarrow x^3 > 1 \Longrightarrow x > 1$$

Otherwise, when $x < 0$ we have

$$x^2 < \frac{1}{x} \Longrightarrow x^3 < 1 \Longrightarrow x < 1$$

Therefore $(x \neq 0) \wedge ((x < 1) \vee (x > 1))$. Therefore $\boxed{(x \neq 0) \wedge (x \neq 1)}$.

Page 30. Number 18. Exercise. *Let $a, b \in \mathbf{R}$, and suppose that for every $\epsilon > 0$ we have $a \leq b + \epsilon$. Show that $a \leq b$.*

Suppose $a, b \in \mathbf{R}$ and $a \leq b + \epsilon$ for every $\epsilon > 0$. Then $a - b \leq \epsilon$. We need to show that $a \leq b$. To do this we will show that $a > b$ is absurd. Suppose $a > b$. Then $a - b > 0$. Then, since we know that $1 > \frac{1}{2}$,

$$1 > \frac{1}{2} \Longrightarrow (a-b) > \frac{(a-b)}{2}$$

But since $a - b$ and $\frac{1}{2}$ are both positive, therefore $\frac{a-b}{2}$ is positive, so that when $\epsilon = \frac{a-b}{2}$ this contradicts $a - b \leq \epsilon$. Therefore it cannot be true that $a > b$.

Therefore $a \leq b$. **Q.E.D.**

Page 30. Number 23. Exercise. *If* $a > 0$, $b > 0$ *and* $n \in \mathbf{N}$, *show that* $a < b$ *if and only if* $a^n < b^n$.

Suppose $a > 0$, $b > 0$, $a, b \in \mathbf{R}$, and $n \in \mathbf{N}$. We need to show both that $a < b \Longrightarrow a^n < b^n$ and that $a^n < b^n \Longrightarrow a < b$.

To show that $a < b \Longrightarrow a^n < b^n$ for any $n \in \mathbf{N}$ we will use mathematical induction. Suppose $a < b$ and let $P(n)$ be the statement $a^n < b^n$. We will show that $P(1)$ is true, that $P(k) \rightarrow P(k+1)$ for any $k \in \mathbf{N}$ and hence that $P(n)$ is true for any $n \in \mathbf{N}$. Then it will follow that $a < b \Longrightarrow a^n < b^n$ for any $n \in \mathbf{N}$.

$P(1)$ is obviously true since $a^1 < b^1 \iff a < b$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then $a^k < b^k$. Now,

$$\begin{aligned} a^k < b^k &\Longrightarrow a^k \cdot a < b^k \cdot a \quad \text{since } a > 0 \\ &\Longrightarrow a^{k+1} < b^k \cdot a \end{aligned}$$

And since $b^k > 0$ follows from $b > 0$ by repeated multiplication by b, we also have that

$$\begin{aligned} a < b &\Longrightarrow b^k \cdot a < b^k \cdot b \\ &\Longrightarrow b^k \cdot a < b^{k+1} \end{aligned}$$

Therefore $a^{k+1} < b^k \cdot a < b^{k+1}$. Therefore $P(k+1)$ is true. Therefore $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $a < b \Longrightarrow a^n < b^n$ for any $n \in \mathbf{N}$.

It remains to show that $a^n < b^n \Longrightarrow a < b$ for all $n \in \mathbf{N}$. Suppose $a^n < b^n$ for some $n \in \mathbf{N}$. Now we must have either $a < b$, $a = b$, or $a > b$ by the Trichotomy property. We need to show that $a < b$. To do this, we will show by contradiction that it cannot be true that $a = b$ or that $a > b$. First suppose that $a = b$. Then $a^n = b^n$, which contradicts our premise that $a^n < b^n$. Now suppose that $a > b$. Then $a^n > b^n$ as shown above (with the letters a and b transposed). This again contradicts our premise that $a^n < b^n$. Therefore it must be that $a^n < b^n \Longrightarrow a < b$.

Therefore if $a > 0$, $b > 0$, and $n \in \mathbf{N}$, then $a < b \iff a^n < b^n$. **Q.E.D.**