Real Analysis: Chapter 1.3

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Page 21, Number 3. Exercise. *Let* $S := \{1, 2\}$ *and* $T := \{a, b, c\}$.

(a) *Determine the number of different injections from* S *into* T.

The element 1 in S can be mapped to any of the 3 elements in T, and for each of these choices there are 2 remaining elements in T to which the element 2 in S can be mapped such that the mapping is injective. Hence, there are $\boxed{3 \cdot 2 = 6}$ different injections from S into T.

(b) *Determine the number of different surjections from* T *onto* S.

There are 2^3 possible mappings of any kind from T to S, only 2 of which are not surjective (either when a, b, and c all map to 1 or when they all map to 2). Hence there are $\boxed{2^3 - 2 = 6}$ different surjections from T onto S.

Page 21, Number 4. Exercise. *Exhibit a bijection between* $\mathbf{N}$ *and the set of all odd integers greater than* 13.

Let S be the set of all odd integers greater than 13. Consider the mapping $\alpha : \mathbf{N} \to S$ defined by $\alpha(x) := 2x + 13$ for all $x \in \mathbf{N}$. We claim that α is a bijection. To show that α is a bijection, we will show that α is both injective and surjective.

To see that α is injective, suppose $\alpha(x_1) = \alpha(x_2)$ for some $x_1, x_2 \in \mathbf{N}$. Then,

$$\alpha(x_1) = \alpha(x_2) \Longrightarrow 2x_1 + 13 = 2x_2 + 13 \Longrightarrow x_1 = x_2$$

and hence α is injective.

To see that α is surjective, suppose y is an odd integer greater than 13. We need to show that there exists an $x \in \mathbf{N}$ such that $\alpha(x) = y$. We claim that $x = \frac{y-13}{2}$ satisfies $x \in \mathbf{N}$ and $\alpha(x) = y$. We have $x \in \mathbf{N}$ since $y - 13 > 0$ and $y - 13$ is even. Finally,

$$\alpha(x) = \alpha\left(\frac{y-13}{2}\right) = 2\left(\frac{y-13}{2}\right) + 13 = y$$

Therefore $x \in \mathbf{N}$ and $\alpha(x) = y$. Therefore α is surjective.

Therefore α is a bijection between $\mathbf{N}$ and S. **Q.E.D.**

Page 21, Number 5. Exercise. *Give an explicit definition of the bijection* f *from* $\mathbf{N}$ *onto* $\mathbf{Z}$ *described in Example 1.3.7(b).*

The mapping $f : \mathbf{N} \to \mathbf{Z}$ may be defined by:

$$f(x) = \begin{cases} \frac{1}{2}x, & \text{if } x \text{ is even} \\ -\frac{1}{2}(x-1), & \text{if } x \text{ is odd} \end{cases}$$

Page 21, Number 8. Exercise. *Give an example of a countable collection of finite sets whose union is not finite.*

The collection of sets $A_1 = \{\emptyset\}$, $A_n = \{A_{n-1}\}$ for $n > 1$, $n \in \mathbf{N}$ is a countable collection of finite sets (since each set has 1 element) whose union $\bigcup_{n=1}^{\infty} A_n$ is not finite (since $\forall x, y \in \mathbf{N}(x \neq y \rightarrow A_x \cap A_y = \emptyset)$).

Page 21, Number 11. Exercise. *Use Mathematical Induction to prove that if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements.*

Let $Q(n)$ be the statement "If the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements". We will show that $Q(1)$ is true and that $Q(k) \rightarrow Q(k+1)$ for any $k \in \mathbf{N}$ and hence, by the principle of Mathematical Induction, that $Q(n)$ is true for all $n \in \mathbf{N}$.

$Q(1)$ is equivalent to the statement "If the set S has 1 element, then $\mathcal{P}(S)$ has 2^1 elements". To see that $Q(1)$ is true, let S be any set with 1 element. Then $S = \{e_1\}$ for some arbitrary element e_1. Now, the set of all subsets of S, $\mathcal{P}(S) = \{\emptyset, \{e_1\}\}$, has 2 elements. Therefore $Q(1)$ is true.

Now suppose $Q(k)$ is true for some $k \in \mathbf{N}$. This means that if the set S has k elements, then $\mathcal{P}(S)$ has 2^k elements. Let S' be any set with $k+1$ elements. We must show that $\mathcal{P}(S')$ has 2^{k+1} elements. Since $k+1 > 1$, there exists a subset X of S' with 1 element x, and so according to Theorem 1.3.4(b) $S' \setminus X$ is a set with $(k+1) - 1 = k$ elements. Let $A = \mathcal{P}(S' \setminus X)$. Then A has 2^k elements. Furthermore, A is the set of all subsets of S' that do not contain the element x. Let $B = \{a \cup X : a \in A\}$. Then $|B| = |A|$ and $A \cap B = \emptyset$. Therefore, according to Theorem 1.3.4(a), $|A \cup B| = |A| + |B| = 2^k + 2^k = 2^{k+1}$. Furthermore, B is the set of all subsets of S' that do contain the element x. To see this, let s be any subset of S' that contains the element x. Then $s \setminus X \subseteq S' \setminus X$ and hence $s \in A$. Therefore, $(s \setminus X) \cup X \in B$ and so $s \in B$.

Now, for any subset s of S', if $x \notin s$, then $s \in A$ but if $x \in s$ then $s \in B$, hence $\mathcal{P}(S') \subseteq A \cup B$. Conversely, there are no elements of A or B that are not subsets of S', hence $A \cup B \subseteq \mathcal{P}(S')$. Therefore $\mathcal{P}(S') = A \cup B$. Therefore $|\mathcal{P}(S')| = |A \cup B| = 2^{k+1}$. Therefore $Q(k) \rightarrow Q(k+1)$.

Therefore, according to the principle of Mathematical Induction, since $Q(1)$ is true and $Q(k) \rightarrow Q(k+1)$ for any $k \in \mathbf{N}$, this implies that $Q(n)$ is true for all $n \in \mathbf{N}$.

Therefore if the set S has n elements, then $\mathcal{P}(S)$ has 2^n elements. **Q.E.D.**