Real Analysis: Chapter 1.2

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Page 15, Number 2. Exercise. *Prove that* $1^3 + 2^3 + \cdots + n^3 = [\frac{1}{2}n(n+1)]^2$ *for all* $n \in \mathbf{N}$.

Let $P(n)$ be the statement $1^3 + 2^3 + \cdots + n^3 = [\frac{1}{2}n(n+1)]^2$. We need to show that $P(n)$ is true for all $n \in \mathbf{N}$. To do this, we shall use the principle of mathematical induction. Hence, we will show that $P(1)$ is true and that $P(k) \to P(k+1)$ for any $k \in \mathbf{N}$.

Now, $P(1)$ is true since $P(1) \iff 1^3 = [\frac{1}{2}(1+1)]^2 = 1$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then,

$$\begin{aligned}
P(k) &\implies 1^3 + 2^3 + \cdots + k^3 = \left[\frac{1}{2}k(k+1)\right]^2 \\
&\implies 1^3 + 2^3 + \cdots + k^3 + (k+1)^3 = \left[\frac{1}{2}k(k+1)\right]^2 + (k+1)^3 \\
&\qquad = (k+1)^2 \left[\left(\frac{1}{2}k\right)^2 + k + 1\right] \\
&\qquad = \left[\frac{1}{2}(k+1)\right]^2 (k^2 + 4k + 4) \\
&\qquad = \left[\frac{1}{2}(k+1)(k+2)\right]^2 \\
&\implies P(k+1)
\end{aligned}$$

Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(1) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $1^3 + 2^3 + \cdots + n^3 = [\frac{1}{2}n(n+1)]^2$ for all $n \in \mathbf{N}$. **Q.E.D.**

Page 16, Number 6. Exercise. *Prove that $n^3 + 5n$ is divisible by 6 for all $n \in \mathbf{N}$.*

Let $f(n) = n^3 + 5n$ and let $P(n)$ be the statement $6|f(n)$. We need to prove that $P(n)$ is true for all $n \in \mathbf{N}$. To do this, we shall use mathematical induction. Hence, we will show that $P(1)$ is true and that $P(k) \to P(k+1)$ for any $k \in \mathbf{N}$.

Now, $P(1)$ is true since $P(1) \iff 6|1^3 + 5$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then $6|f(k)$. Now let $x = f(k+1) - f(k)$ so that $P(k+1) \iff 6|f(k) + x$. Hence we have that if $6|x$ then $P(k+1)$ is true. Solving for x:

$$\begin{aligned} x &= f(k+1) - f(k) \\ &= (k+1)^3 + 5(k+1) - (k^3 + 5k) \\ &= k^3 + 3k^2 + 3k + 1 + 5k + 5 - k^3 - 5k \\ &= 3k^2 + 3k + 6 \\ &= 3k(k+1) + 6 \end{aligned}$$

Either k or $k+1$ must be divisible by 2, consequently $6|3k(k+1)$ and so $6|x$. Therefore $P(k+1)$ is true. Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(1) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $n^3 + 5n$ is divisible by 6 for all $n \in \mathbf{N}$. **Q.E.D.**

Page 16, Number 7. Exercise. *Prove that $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbf{N}$.*

Let $P(n)$ be the statement $8|5^{2n} - 1$. We need to prove that $P(n)$ is true for all $n \in \mathbf{N}$. To do this, we shall use mathematical induction. Hence, we will show that $P(1)$ is true and that $P(k) \to P(k+1)$ for any $k \in \mathbf{N}$.

Now, $P(1)$ is true since $P(1) \iff 8|5^2 - 1 \iff 8|24$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then,

$$\begin{aligned} P(k) &\Longrightarrow 8|5^{2k} - 1 \\ &\Longrightarrow 5^{2k} \equiv 1 \pmod 8 \\ &\Longrightarrow 5^{2k}5^2 \equiv 5^2 \pmod 8 \\ &\Longrightarrow 5^{2k+2} \equiv 1 \pmod 8 \\ &\Longrightarrow 5^{2(k+1)} \equiv 1 \pmod 8 \\ &\Longrightarrow 8|5^{2(k+1)} - 1 \\ &\Longrightarrow P(k+1) \end{aligned}$$

Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(1) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $5^{2n} - 1$ is divisible by 8 for all $n \in \mathbf{N}$. **Q.E.D.**

Page 16, Number 11. Exercise. *Conjecture a formula for the sum of the first n odd natural numbers $1 + 3 + \cdots + (2n - 1)$ and prove your formula by using Mathematical Induction.*

Let $P(n)$ be the statement $1+3+\cdots+(2n-1) = n^2$. We will show that $P(n)$ is true for all $n \in \mathbf{N}$. To do this, we shall use mathematical induction. Hence, we will show that $P(1)$ is true and that $P(k) \to P(k+1)$ for any $k \in \mathbf{N}$.

Now, $P(1)$ is true since $P(1) \iff 1 = 1^2$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then,

$$\begin{aligned} P(k) &\Longrightarrow 1 + 3 + \cdots + (2k-1) = k^2 \\ &\Longrightarrow 1 + 3 + \cdots + (2k-1) + (2k+1) = k^2 + (2k+1) = (k+1)^2 \\ &\Longrightarrow P(k+1) \end{aligned}$$

Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(1) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $1 + 3 + \cdots + (2n-1) = n^2$ for all $n \in \mathbf{N}$. **Q.E.D.**

Page 16, Number 14. Exercise. *Prove that $2^n < n!$ for all $n \geq 4$, $n \in \mathbf{N}$.*

Let $P(n)$ be the statement $2^n < n!$ We will show that $P(n)$ is true for all $n \geq 4$, $n \in \mathbf{N}$. To do this, we shall use mathematical induction. Hence, we will show that $P(4)$ is true and that $P(k) \to P(k+1)$ for any $k \geq 4$, $k \in \mathbf{N}$.

Now, $P(4)$ is true since $P(4) \iff 2^4 < 4! \iff 16 < 24$.

Suppose $P(k)$ is true for some $k \geq 4$, $k \in \mathbf{N}$. Then,

$$\begin{aligned} P(k) &\Longrightarrow 2^k < k! \\ &\Longrightarrow 2^k \cdot 2 < (k!)(k+1) \text{ since } 2 < k+1 \text{ for } k \geq 4 \\ &\Longrightarrow 2^{k+1} < (k+1)! \\ &\Longrightarrow P(k+1) \end{aligned}$$

Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(4) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \geq 4$, $n \in \mathbf{N}$. Therefore $2^n < n!$ for all $n \geq 4$, $n \in \mathbf{N}$. **Q.E.D.**

Page 16, Number 18. Exercise. *Prove that* $1/\sqrt{1} + 1/\sqrt{2} + \cdots + 1/\sqrt{n} \geq \sqrt{n}$ *for all* $n \in \mathbf{N}$.

Let $P(n)$ be the statement $1/\sqrt{1}+1/\sqrt{2}+\cdots+1/\sqrt{n} \geq \sqrt{n}$. We will show that $P(n)$ is true for all $n \in \mathbf{N}$. To do this, we shall use mathematical induction. Hence, we will show that $P(1)$ is true and that $P(k) \to P(k+1)$ for any $k \in \mathbf{N}$.

Now, $P(1)$ is true since $P(1) \iff 1/\sqrt{1} \geq \sqrt{1}$.

Suppose $P(k)$ is true for some $k \in \mathbf{N}$. Then,

$$\begin{aligned}
P(k) &\implies \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} \geq \sqrt{k} \\
&\implies \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} \geq \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\
&\qquad \geq \frac{\sqrt{k}\sqrt{k}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1} \\
&\implies P(k+1)
\end{aligned}$$

Therefore $P(k) \to P(k+1)$.

Therefore, according to the principle of mathematical induction, since $P(1) \wedge (P(k) \to P(k+1))$, consequently $P(n)$ is true for all $n \in \mathbf{N}$. Therefore $1/\sqrt{1}+1/\sqrt{2}+\cdots+1/\sqrt{n} \geq \sqrt{n}$ for all $n \in \mathbf{N}$. **Q.E.D.**