

Abstract Algebra: Section 9

David Joseph Stith

9.18 Problem. For sets S and T , let $S \sim T$ mean that there is an invertible mapping of S onto T . Prove that \sim is reflexive, symmetric, and transitive.

Suppose that for sets S and T , $S \sim T$ if and only if there is an invertible mapping of S onto T . We need to show that \sim is reflexive, symmetric, and transitive. To do this, we will show that

- (i) for any set S , $S \sim S$.
- (ii) for any sets S and T , $S \sim T \implies T \sim S$.
- (iii) for any sets S , T , and U , $S \sim T \wedge T \sim U \implies S \sim U$.

We proceed as follows.

- (i) Suppose S is a set. We know there exists an identity mapping ι_S that maps every element of S to itself. Furthermore, ι_S is invertible since it is both one-to-one and onto. Therefore there is an invertible mapping from S onto S . Therefore $S \sim S$. Therefore \sim is reflexive.
- (ii) Suppose S and T are sets such that $S \sim T$. Then there is an invertible mapping from S onto T . Let α be an invertible mapping from S onto T . Then α^{-1} exists since α is invertible. Then, since $\alpha \circ \alpha^{-1} = \iota$ and $\alpha^{-1} \circ \alpha = \iota$, we know that α is the inverse of α^{-1} . Therefore α^{-1} is an invertible mapping from T onto S . Therefore $T \sim S$. Therefore \sim is symmetric.
- (iii) Suppose S , T , and U are sets such that $S \sim T \wedge T \sim U$. Then there is an invertible mapping from S onto T , and an invertible mapping from T onto U . Let α be an invertible mapping from S onto T , and let β be an invertible mapping from T onto U . Then $\beta \circ \alpha$ exists and is a mapping from S to U . We need to show that $\beta \circ \alpha$ is invertible and hence that $S \sim U$. To do this we will show that $\beta \circ \alpha$ is both one-to-one and onto. We know that α is both one-to-one and onto since it is invertible. Likewise, we know that β is both one-to-one and onto since it is invertible. Then, since α and β are one-to-one, $\beta \circ \alpha$ is one-to-one. And since α and β are onto, $\beta \circ \alpha$ is onto. Therefore $\beta \circ \alpha$ is both one-to-one and onto, and hence is invertible. Therefore $S \sim U$. Therefore \sim is transitive.

Therefore \sim is reflexive, symmetric, and transitive.

Q.E.D.