Abstract Algebra: Section 5

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5.22 Problem. *Prove that if G is a group, $a \in G$, and $a * b = b$ for some $b \in G$, then a must be the identity element of G.*

Suppose that $(G, *)$ is a group, $a \in G$, and $a * b = b$ for some $b \in G$. Then, by the definition of a group, the following are true:

(Associativity)

$x * (y * z) = (x * y) * z$ for all $x, y, z \in G$.

(Existence of an identity element)

There is an element $e \in G$ such that $x * e = e * x = x$ for each $x \in G$.

(Existence of inverse elements)

For each $x \in G$ there is an element $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$.

We must show that $a = e$. To do this, we use all three of the above properties as follows:

$$\begin{aligned} a &= a * e && \textit{(Existence of an identity element)} \\ &= a * (b * b^{-1}) && \textit{(Existence of inverse elements)} \\ &= (a * b) * b^{-1} && \textit{(Associativity)} \\ &= b * b^{-1} && (a * b = b \textit{ by our supposition)} \\ &= e \end{aligned}$$

Therefore $a = e$ so that a is the identity element of G.

Therefore if $(G, *)$ is a group, $a \in G$, and $a * b = b$ for some $b \in G$, then a must be the identity element of G. **Q.E.D.**