Abstract Algebra: Section 21

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21.20 Problem. *Prove that $N \triangleleft G$ iff $g^{-1}ng \in N$ for all $n \in N$ and $g \in G$.*

We need to show that both of the following are true:

(i) $N \triangleleft G \implies g^{-1}ng \in N$ for all $n \in N$ and $g \in G$.

(ii) $g^{-1}ng \in N$ for all $n \in N$ and $g \in G \implies N \triangleleft G$.

We proceed as follows:

(i) Suppose $N \triangleleft G$. Then $hnh^{-1} \in N$ for all $n \in N$ and $h \in G$. Suppose $g \in G$. Then since G is a group, $g^{-1} \in G$ so that letting $h = g^{-1}$ we have

$$\begin{aligned} N \triangleleft G &\implies hnh^{-1} \in N \text{ for all } n \in N \\ &\implies g^{-1}n(g^{-1})^{-1} \in N \\ &\implies g^{-1}ng \in N \quad \text{by Theorem 14.1(d)} \end{aligned}$$

Therefore $N \triangleleft G \implies g^{-1}ng \in N$ for all $n \in N$ and $g \in G$.

(ii) Suppose $g^{-1}ng \in N$ for all $n \in N$ and $g \in G$. We need to show that $N \triangleleft G$. To do this we will show that $hnh^{-1} \in N$ for all $n \in N$ and $h \in G$. Suppose $h \in G$. Then since G is a group, $h^{-1} \in G$ so that letting $g = h^{-1}$ we have

$$\begin{aligned} g^{-1}ng \in N &\implies (h^{-1})^{-1}nh^{-1} \in N \text{ for all } n \in N \\ &\implies hnh^{-1} \in N \quad \text{by Theorem 14.1(d)} \\ &\implies N \triangleleft G \end{aligned}$$

Therefore $g^{-1}ng \in N$ for all $n \in N$ and $g \in G \implies N \triangleleft G$.

Therefore $N \triangleleft G$ iff $g^{-1}ng \in N$ for all $n \in N$ and $g \in G$. **Q.E.D.**