Abstract Algebra: Section 2

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2.7 Problem. *Prove that if $\alpha : S \to T$, then $\alpha \circ \iota_S = \alpha$ and $\iota_T \circ \alpha = \alpha$.*

Suppose that

$$\alpha : S \to T,$$
$$\iota_S : S \to S, \quad \forall x \in S(\iota_S(x) = x)$$
$$\iota_T : T \to T, \quad \forall x \in T(\iota_T(x) = x)$$

We need to show that $\alpha \circ \iota_S = \alpha$ and $\iota_T \circ \alpha = \alpha$. To do this, we will show that $\forall x \in S$, $(\alpha \circ \iota_S)(x) = \alpha(x)$ and $(\iota_T \circ \alpha)(x) = \alpha(x)$.

For the first of these, we have

$$(\alpha \circ \iota_S)(x) = \alpha(\iota_S(x)) = \alpha(x)$$

Therefore $(\alpha \circ \iota_S)(x) = \alpha(x)$.

For the second, we have

$$(\iota_T \circ \alpha)(x) = \iota_T(\alpha(x)) = \alpha(x)$$

Therefore $(\iota_T \circ \alpha)(x) = \alpha(x)$.

Therefore $\alpha \circ \iota_S = \alpha$ and $\iota_T \circ \alpha = \alpha$. **Q.E.D.**

2.27 Problem. *Assume that $\alpha : S \to T$ and $\beta : T \to U$. Use Theorems 2.1 and 2.2 to prove each of the following statements.*

(a) *If α and β are invertible, then $\beta \circ \alpha$ is invertible.*

Suppose α and β are invertible. We need to show that $\beta \circ \alpha$ is invertible. To do this, we will use Theorem 2.1 and Theorem 2.2 as follows:

By Theorem 2.2, since α is invertible, therefore α is both one-to-one and onto. Likewise, since β is invertible, therefore β is both one-to-one and onto.

Then, by Theorem 2.1(c), since α and β are one-to-one, therefore $\beta \circ \alpha$ is one-to-one. Also, by Theorem 2.1(a), since α and β are onto, therefore $\beta \circ \alpha$ is onto.

So, by Theorem 2.2, since $\beta \circ \alpha$ is both one-to-one and onto, therefore $\beta \circ \alpha$ is invertible.

Therefore, if α and β are invertible, then $\beta \circ \alpha$ is invertible. **Q.E.D.**

(b) *If $\beta \circ \alpha$ is invertible, then β is onto and α is one-to-one.*

Suppose $\beta \circ \alpha$ is invertible. We need to show that β is onto and α is one-to-one. To do this, we will use Theorem 2.1 and Theorem 2.2 as follows:

By Theorem 2.2, since $\beta \circ \alpha$ is invertible, therefore $\beta \circ \alpha$ is both one-to-one and onto.

Then, by Theorem 2.1(b), since $\beta \circ \alpha$ is onto, therefore β is onto.

And, by Theorem 2.1(d), since $\beta \circ \alpha$ is one-to-one, therefore α is one-to-one.

Therefore, if $\beta \circ \alpha$ is invertible, then β is onto and α is one-to-one. **Q.E.D.**