

Abstract Algebra: Section 18

David Joseph Stith

We will use the following previously established theorem in the proof that follows:

2.1 Theorem. Assume that $\alpha : S \rightarrow T$ and $\beta : T \rightarrow U$.

- (a) If α and β are onto, then $\beta \circ \alpha$ is onto.
- (c) If α and β are one-to-one, then $\beta \circ \alpha$ is one-to-one.

18.9 Problem. Prove that if G , H , and K are groups, and $\theta : G \rightarrow H$ and $\phi : H \rightarrow K$ are isomorphisms, then $\phi \circ \theta : G \rightarrow K$ is an isomorphism.

Suppose G , H , and K are groups and $\theta : G \rightarrow H$ and $\phi : H \rightarrow K$ are isomorphisms. We need to show that $\phi \circ \theta : G \rightarrow K$ is an isomorphism. To do this we will show that

- (i) $\phi \circ \theta$ is one-to-one and onto, and
- (ii) $(\phi \circ \theta)(ab) = (\phi \circ \theta)(a) \cdot (\phi \circ \theta)(b)$ for all $a, b \in G$.

It will then follow that $\phi \circ \theta$ is an isomorphism.

We proceed as follows:

- (i) Since θ and ϕ are both isomorphisms, therefore they are both one-to-one, therefore by Theorem 2.1(c) $\phi \circ \theta$ is one-to-one. Likewise, since θ and ϕ are both onto, therefore by Theorem 2.1(a) $\phi \circ \theta$ is onto. Therefore $\phi \circ \theta$ is one-to-one and onto.
- (ii) Suppose $a, b \in G$. We have

$$\begin{aligned}(\phi \circ \theta)(ab) &= \phi(\theta(ab)) \\ &= \phi(\theta(a) \cdot \theta(b)) \text{ since } \theta \text{ is an isomorphism} \\ &= \phi(\theta(a)) \cdot \phi(\theta(b)) \text{ since } \phi \text{ is an isomorphism} \\ &= (\phi \circ \theta)(a) \cdot (\phi \circ \theta)(b)\end{aligned}$$

Therefore $(\phi \circ \theta)(ab) = (\phi \circ \theta)(a) \cdot (\phi \circ \theta)(b)$ for all $a, b \in G$.

Therefore since $\phi \circ \theta$ satisfies both (i) and (ii), $\phi \circ \theta : G \rightarrow K$ is an isomorphism.

Q.E.D.