# Abstract Algebra: Section 17

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**17.25 Problem.** *Prove that if $H$ is a subgroup of $G$, $[G : H] = 2$, $a, b \in G$, $a \notin H$, and $b \notin H$, then $ab \in H$.*

Suppose that $H$ is a subgroup of $G$, $[G : H] = 2$, $a, b \in G$, $a \notin H$, and $b \notin H$. Then since there are 2 right cosets of $H$ in $G$, one of which is $H$, let $J$ be the other right coset of $H$ in $G$ so that every element of $G$ is either in $H$ or in $J$. We need to show that $ab \in H$. To do this we will show that for every $j \in J$, the unique solution of $xb = j$ is such that $x \neq a$. It will then follow that $ab \notin J$ and therefore $ab \in H$.

Now since $b \notin H$ therefore $b \in J$ and hence $Hb = J$. Therefore for every $j \in J$, there is an $h \in H$ such that $hb = j$. Then since $a \notin H$, $h \neq a$. Therefore $ab \notin J$. Therefore $ab \in H$. **Q.E.D.**

**17.27 Problem.** *Prove that if $A$ and $B$ are finite subgroups of a group $G$, and $|A|$ and $|B|$ have no common divisor greater than 1, then $A \cap B = \{e\}$.*

Suppose $A$ and $B$ are finite subgroups of a group $G$ and $\gcd(|A|, |B|) = 1$. We need to show that $A \cap B = \{e\}$. To do this we will show that no element other than $e$ can exist in both $A$ and $B$. Hence it will follow from the fact that $e \in A$ and $e \in B$, since $A$ and $B$ are groups, that $A \cap B = \{e\}$.

Suppose to the contrary that there exists an element $x \neq e$ such that $x \in A$ and $x \in B$ and hence that $\{e\} \subset A \cap B$. Then $o(x) > 1$ since $x \neq e$, and $\langle x \rangle \in A$ and $\langle x \rangle \in B$ with $o(x) = |\langle x \rangle|$ so that by Lagrange's Theorem $o(x)$ divides both $|A|$ and $|B|$. But then $\gcd(|A|, |B|) \geq o(x) > 1$ contradicts our original supposition that $\gcd(|A|, |B|) = 1$. Hence it cannot be true that $x$ exists in both $A$ and $B$. Therefore $A \cap B = \{e\}$. **Q.E.D.**