Abstract Algebra: Section 14

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14.25 Problem. *Prove that a group G is Abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.*

We need to show that both of the following are true:

(i) If G is Abelian then $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

(ii) If $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$ then G is Abelian.

We proceed as follows:

(i) Suppose G is Abelian and $a, b \in G$. Then $(ab)^{-1} = (ba)^{-1}$. And by Theorem 14.1(e) we know that $(ba)^{-1} = a^{-1}b^{-1}$. Therefore $(ab)^{-1} = a^{-1}b^{-1}$.
Therefore it is true that if G is Abelian then $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$.

(ii) Now suppose that $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. By Theorem 14.1(e) we know that $(ba)^{-1} = a^{-1}b^{-1}$. Therefore we have

$$\begin{aligned}(ab)^{-1} = (ba)^{-1} &\Longrightarrow (ab)^{-1}(ab) = (ba)^{-1}(ab)\\ &\Longrightarrow e = (ba)^{-1}(ab)\\ &\Longrightarrow (ba)(e) = (ba)(ba)^{-1}(ab)\\ &\Longrightarrow (ba)(e) = (e)(ab)\\ &\Longrightarrow ba = ab\end{aligned}$$

Therefore G is Abelian.
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Therefore, since both (i) and (ii) above are true, we may conclude that a group G is Abelian iff $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in G$. **Q.E.D.**

14.34 Problem. *Prove that if a group G has no subgroup other than G and $\{e\}$, then G is cyclic.*

Suppose G is a group such that G and $\{e\}$ are the only subgroups of G. We need to show that G is cyclic. To do this, we will show that there exists an $a \in G$ such that $\langle a \rangle = G$.

Now, if $G = \{e\}$ then G is cyclic since $G = \langle e \rangle$.

Otherwise, let $a \in G$ such that $a \neq e$. Then $\langle a \rangle$ is a subgroup of G by Theorem 14.2. Therefore a must be either $\{e\}$ or G since these are the only subgroups of G. But $\langle a \rangle \neq \{e\}$ since $a \neq e$. Therefore $\langle a \rangle = G$.

Therefore by definition G is cyclic. **Q.E.D.**