Mathematical Methods in the Physical Sciences

Chapter 3 Homework

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Section 3.

4. *Evaluate the determinant.*

$$\begin{vmatrix} -2 & 4 & 7 & 3 \\ 8 & 2 & -9 & 5 \\ -4 & 6 & 8 & 4 \\ 2 & -9 & 3 & 8 \end{vmatrix} \xrightarrow[\substack{R_3\to R_3-2R_1 \\ R_4\to R_4+R_1}]{R_2\to R_2+4R_1} \begin{vmatrix} -2 & 4 & 7 & 3 \\ 0 & 18 & 19 & 17 \\ 0 & -2 & -6 & -2 \\ 0 & -5 & 10 & 11 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 18 & 19 & 17 \\ -2 & -6 & -2 \\ -5 & 10 & 11 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 18 & 19 & 17 \\ 1 & 3 & 1 \\ -5 & 10 & 11 \end{vmatrix}$$

$$= 4 \left[18 \begin{vmatrix} 3 & 1 \\ 10 & 11 \end{vmatrix} - \begin{vmatrix} 19 & 17 \\ 10 & 11 \end{vmatrix} - 5 \begin{vmatrix} 19 & 17 \\ 3 & 1 \end{vmatrix} \right]$$

$$= 4\left[18(33-10) - (209-170) - 5(19-51)\right]$$

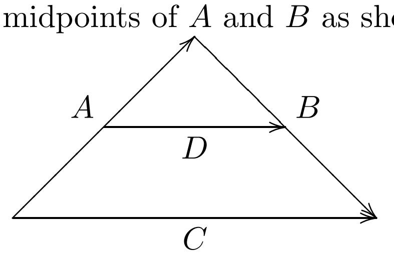
$$= 4[414 - 39 + 160]$$

$$= \boxed{2140}$$

Section 4.

4. *Use vectors to prove the following theorem from geometry: The line segment joining the midpoints of two sides of any triangle is parallel to the third side and half its length.*

Suppose A, B, and C are the sides of a triangle. Let D be a line segment joining the midpoints of A and B as shown. Then,



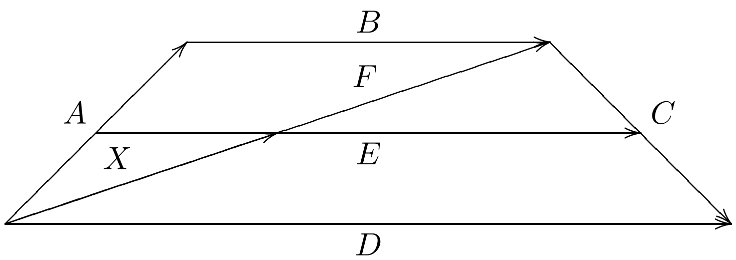
$$C = A + B$$
$$D = \tfrac{1}{2}A + \tfrac{1}{2}B = \tfrac{1}{2}(A+B) = \tfrac{1}{2}C$$

Therefore, since $D = \frac{1}{2}C$, D is parallel to C and half its length.

Therefore the line segment joining the midpoints of two sides of any triangle is parallel to the third side and half its length. **Q.E.D.**

8. *Use vectors to prove the following theorem from geometry: The median of a trapezoid (four-sided figure with just two parallel sides) means the line joining the midpoints of the two nonparallel sides. Prove that the median bisects both diagonals; that the median is parallel to the two parallel bases and equal to half the sum of their lengths.*

Suppose A, B, C, D are the sides of a trapezoid, with $B \parallel D$. Let E be the median of the trapezoid and let F be a diagonal. Let X be the portion of F that terminates at the intersection of E and F.



First we must show that E is parallel to B and D and equal to half the sum of their lengths. To do this, we will show that $E = \frac{1}{2}(B + D)$ and hence that $E \parallel B$ and $E \parallel D$ since $B \parallel D$. We have,

$$\begin{aligned} D &= A + B + C \\ E &= \frac{1}{2}A + B + \frac{1}{2}C \\ &= \frac{1}{2}(A + 2B + C) \\ &= \frac{1}{2}(B + D) \end{aligned}$$

Therefore E is parallel to B and D and equal to half the sum of their lengths.

Now we must show that E bisects both diagonals. To do this for the diagonal shown, F, we will show that $X = \frac{1}{2}F$. We have,

$$\begin{aligned} F &= A + B \\ X &= \frac{1}{2}A + k_1 B \text{ for some constant } k_1 \text{ since } E \parallel B \\ X &= k_2 F \text{ for some constant } k_2 \text{ since } X \parallel F \\ &= k_2 A + k_2 B \end{aligned}$$

Therefore, since these two representation of X are equal, $\frac{1}{2}A + k_1 B = k_2 A + k_2 B$. Hence,

$$\left(\frac{1}{2} - k_2\right) A = (k_2 - k_1)B$$

Then, since A is not parallel to B, this equality is possible only when $\frac{1}{2} - k_2 = 0$ and $k_2 - k_1 = 0$. Therefore, $k_1 = k_2 = \frac{1}{2}$ and thus $X = \frac{1}{2}F$. Therefore E bisects F, and by applying the same argument to the other diagonal it is easy to see that E bisects both diagonals.

Therefore the median bisects both diagonals, is parallel to the two parallel bases, and is equal to half the sum of their lengths. **Q.E.D.**

Section 5.

21. *Find the angle between the given planes:* $2x + 6y - 3z = 10$ *and* $5x + 2y - z = 12$.

The angle between these two planes is equivalent to the angle between the normal vectors of the two planes. Hence, we must find the angle between $\vec{N_1}$ and $\vec{N_2}$ given by

$$\vec{N_1} = 2\vec{i} + 6\vec{j} - 3\vec{k} \text{ and } \vec{N_2} = 5\vec{i} + 2\vec{j} - \vec{k}$$

where $\vec{i}$, $\vec{j}$, and $\vec{k}$ are unit vectors in the direction of x, y, and z respectively.

To do this, we solve for θ using the relation:

$$\vec{N_1} \cdot \vec{N_2} = |\vec{N_1}||\vec{N_2}| \cos\theta$$

Hence,

$$\begin{aligned}
\cos\theta &= \frac{\vec{N_1} \cdot \vec{N_2}}{|\vec{N_1}||\vec{N_2}|} \\
&= \frac{(2, 6, -3) \cdot (5, 2, -1)}{\sqrt{2^2 + 6^2 + 3^2}\sqrt{5^2 + 2^2 + 1^2}} \\
&= \frac{10 + 12 + 3}{\sqrt{49}\sqrt{30}} \\
&= \frac{25}{7\sqrt{30}}
\end{aligned}$$

Therefore $\boxed{\theta = \cos^{-1}\left(\frac{25}{7\sqrt{30}}\right) \approx 49.3°}$.

39. *Show that the given lines intersect and find the acute angle between them:*

$$\vec{r_1} = 2\vec{j} + \vec{k} + (3\vec{i} - \vec{k})t_1 \text{ and } \vec{r_2} = 7\vec{i} + 2\vec{k} + (2\vec{i} - \vec{j} + \vec{k})t_2$$

We know that the two lines intersect if and only if there exists a t_1 and t_2 such that $r_1 = r_2$. Let $r_1 = x_1\vec{i} + y_1\vec{j} + z_1\vec{k}$ and $r_2 = x_2\vec{i} + y_2\vec{j} + z_2\vec{k}$. Then, r_1 and r_2 intersect if there exists a t_1 and t_2 such that $x_1 = x_2$, $y_1 = y_2$, and $z_1 = z_2$. We have,

$$\begin{aligned}
x_1 &= 3t_1 & x_2 &= 7 + 2t_2 \\
y_1 &= 2 & y_2 &= -t_2 \\
z_1 &= 1 - t_1 & z_2 &= 2 + t_2
\end{aligned}$$

Therefore,

$$\left\{\begin{aligned}
3t_1 &= 7 + 2t_2 \\
2 &= -t_2 \\
1 - t_1 &= 2 + t_2
\end{aligned}\right\}$$

Using the first two of these equations, we have

$$\begin{aligned}
t_2 = -2 &\Longrightarrow 3t_1 = 7 - 4 \\
&\Longrightarrow t_1 = 1
\end{aligned}$$

Substituting these values into the third equation shows that the system is consistent:

$$1 - t_1 = 2 + t_2 \Longrightarrow 1 - 1 = 2 - 2$$

Therefore, when $t_1 = 1$ and $t_2 = -2$, $r_1 = r_2$ and hence the given lines intersect. In particular,

$$\begin{aligned} r_1 &= 2\vec{j} + \vec{k} + (3\vec{i} - \vec{k}) \\ &= 3\vec{i} + 2\vec{j} \\ r_2 &= 7\vec{i} + 2\vec{k} + (2\vec{i} - \vec{j} + \vec{k})(-2) \\ &= 7\vec{i} + 2\vec{k} - 4\vec{i} + 2\vec{j} - 2\vec{k} \\ &= 3\vec{i} + 2\vec{j} \\ &= r_1 \end{aligned}$$

Q.E.D.

To find the angle θ between r_1 and r_2, we find the angle between their parametric vectors, $(3, 0, -1)$ and $(2, -1, 1)$. Hence,

$$\begin{aligned} \cos\theta &= \frac{(3, 0, -1) \cdot (2, -1, 1)}{\sqrt{3^2 + 1^2}\sqrt{2^2 + 1^2 + 1^2}} \\ &= \frac{6 - 1}{\sqrt{10}\sqrt{6}} \\ &= \frac{5}{2\sqrt{15}} \end{aligned}$$

Therefore $\boxed{\theta = \cos^{-1}\left(\frac{5}{2\sqrt{15}}\right) \approx 49.8^\circ}$.

41. *Find the distance between the two given lines.* $r_1 = (4,3,-1) + (1,1,1)t_1$ *and* $r_2 = (4,-1,1) + (1,-2,-1)t_2$

The direction along which the distance should be measured is perpendicular to both r_1 and r_2 and hence is given by the cross product of the parametric vectors:

$$\begin{aligned}(1,1,1)\times(1,-2,-1) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ 1 & -2 & -1 \end{vmatrix} \\ &= \vec{i}(1) - \vec{j}(-2) + \vec{k}(-3) \\ &= (1,2,-3)\end{aligned}$$

Expressing this vector as a unit vector, we have:

$$\frac{(1,2,-3)}{\sqrt{1+2^2+3^2}} = \frac{(1,2,-3)}{\sqrt{14}}$$

Hence the distance between r_1 and r_2 is given by the distance between any point on r_1 and any point on r_2 measured along the vector we found above. Hence we use the dot product with the vector between points at $t_1 = 0$, $t_2 = 0$:

$$\begin{aligned}&\left|[(4,3,-1)-(4,-1,1)]\cdot\frac{(1,2,-3)}{\sqrt{14}}\right| \\ =&\left|\frac{(0,4,-2)\cdot(1,2,-3)}{\sqrt{14}}\right| \\ =&\left|\frac{0+8+6}{\sqrt{14}}\right| \\ =&\boxed{\sqrt{14}}\end{aligned}$$

Section 6.

6. *The Pauli spin matrices in quantum mechanics are*

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Show that $A^2 = B^2 = C^2 =$ a unit matrix. Also show that any two of these matrices anticommute, that is, $AB = -BA$, etc. Show that the commutator of A and B, that is, $AB - BA$, is $2iC$, and similarly for other pairs in cyclic order.

$$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$B^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$$

$$AB = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = -BA$$

$$CA = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$AC = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -CA$$

$$CB = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - CB$$

$$AB - BA = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} = 2iC$$

$$BC - CB = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} = 2iA$$

$$CA - AC = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = 2iB$$

13. *Find the inverse of the given matrix.*

$$\begin{pmatrix} 6 & 9 \\ 3 & 5 \end{pmatrix}$$

The determinant of this matrix is $30 - 27 = 3$ *therefore the inverse is given by:*

$$\begin{aligned}\frac{1}{3}\begin{pmatrix} 5 & -3 \\ -9 & 6 \end{pmatrix}^T &= \frac{1}{3}\begin{pmatrix} 5 & -9 \\ -3 & 6 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} \frac{5}{3} & -3 \\ -1 & 2 \end{pmatrix}}\end{aligned}$$

To show that this is the inverse of the given matrix, we have

$$\begin{pmatrix} 6 & 9 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 5/3 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} 5/3 & -3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 6 & 9 \\ 3 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

17. *Given the matrices*

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 4 & -2 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

(a) *Find* A^{-1}, B^{-1}, $B^{-1}AB$, *and* $B^{-1}A^{-1}B$.

$$\begin{aligned}A^{-1} &= \frac{1}{\det(A)}\begin{pmatrix} \begin{vmatrix} 0 & -1 \\ -2 & 0 \end{vmatrix} & -\begin{vmatrix} 4 & -1 \\ 4 & 0 \end{vmatrix} & \begin{vmatrix} 4 & 0 \\ 4 & -2 \end{vmatrix} \\ -\begin{vmatrix} -1 & 1 \\ -2 & 0 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 4 & -2 \end{vmatrix} \\ \begin{vmatrix} -1 & 1 \\ 0 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 4 & -1 \end{vmatrix} & \begin{vmatrix} 1 & -1 \\ 4 & 0 \end{vmatrix} \end{pmatrix}^T \\ &= \frac{1}{-2+4-8}\begin{pmatrix} -2 & -4 & -8 \\ -2 & -4 & -2 \\ 1 & 5 & 4 \end{pmatrix}^T \\ &= \frac{1}{6}\begin{pmatrix} -2 & -2 & 1 \\ -4 & -4 & 5 \\ -8 & -2 & 4 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 1/3 & 1/3 & -1/6 \\ 2/3 & 2/3 & -5/6 \\ 4/3 & 1/3 & -2/3 \end{pmatrix}}\end{aligned}$$

$$B^{-1} = \frac{1}{\det B} \begin{pmatrix} \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 1 \\ 2 & 2 \end{vmatrix} & \begin{vmatrix} 2 & 1 \\ 2 & 1 \end{vmatrix} \\ -\begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} & \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} & -\begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} & \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix}^T$$

$$= \frac{1}{1+0} \begin{pmatrix} 1 & -2 & 0 \\ 1 & 0 & -1 \\ -1 & 1 & 1 \end{pmatrix}^T$$

$$= \boxed{\begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}}$$

Then,

$$B^{-1}AB = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 4 & 0 & -1 \\ 4 & -2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 2 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix}}$$

$$B^{-1}A^{-1}B = \begin{pmatrix} 1 & 1 & -1 \\ -2 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & -1/6 \\ 2/3 & 2/3 & -5/6 \\ 4/3 & 1/3 & -2/3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1/3 & 2/3 & -1/3 \\ 2/3 & -1/3 & -1/3 \\ 2/3 & -1/3 & 1/6 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 2 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/3 & 1/3 & -1/3 \\ -2/3 & -2/3 & -1/3 \\ 1/3 & -1/6 & 2/3 \end{pmatrix}}$$

(b) *Show that the last two matrices are inverses, that is, that their product is the unit matrix.*

$$(B^{-1}AB)(B^{-1}A^{-1}B) = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix} \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ -2/3 & -2/3 & -1/3 \\ 1/3 & -1/6 & 2/3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$(B^{-1}A^{-1}B)(B^{-1}AB) = \begin{pmatrix} 1/3 & 1/3 & -1/3 \\ -2/3 & -2/3 & -1/3 \\ 1/3 & -1/6 & 2/3 \end{pmatrix} \begin{pmatrix} 3 & 1 & 2 \\ -2 & -2 & -2 \\ -2 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q.E.D.

Section 8.

25. *Find the values of λ such that the following equations have nontrivial solutions, and for each λ, solve the equations.*

$$\left\{ \begin{aligned} -(1+\lambda)x + y + 3z &= 0 \\ x + (2-\lambda)y &= 0 \\ 3x + (2-\lambda)z &= 0 \end{aligned} \right\}$$

These have nontrivial solutions if and only if the determinant of the coefficient matrix is zero. Hence,

$$\begin{aligned} &\begin{vmatrix} -(1+\lambda) & 1 & 3 \\ 1 & 2-\lambda & 0 \\ 3 & 0 & 2-\lambda \end{vmatrix} = 0 \\ \Longrightarrow &3 \begin{vmatrix} 1 & 3 \\ 2-\lambda & 0 \end{vmatrix} + (2-\lambda) \begin{vmatrix} -(1+\lambda) & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \\ \Longrightarrow &3(-3(2-\lambda)) + (2-\lambda)(-(2+\lambda-\lambda^2) - 1) = 0 \\ \Longrightarrow &-18 + 9\lambda + (2-\lambda)(\lambda^2 - \lambda - 3) = 0 \\ \Longrightarrow &-18 + 9\lambda + 2\lambda^2 - 2\lambda - 6 - \lambda^3 + \lambda^2 + 3\lambda = 0 \\ \Longrightarrow &-\lambda^3 + 3\lambda^2 + 10\lambda - 24 = 0 \\ \Longrightarrow &(\lambda-2)(-\lambda^2 + \lambda + 12) = 0 \\ \Longrightarrow &-(\lambda-2)(\lambda+3)(\lambda-4) = 0 \\ \Longrightarrow &\boxed{\lambda \in \{2, -3, 4\}} \end{aligned}$$

Now, when $\lambda = 2$,

$$\begin{pmatrix} -3 & 1 & 3 \\ 1 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow[R_3 \to R_3 - 3R_2]{R_1 \to R_1 - R_3} \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Longrightarrow x = 0 \text{ and } y + 3z = 0$$

$$\Longrightarrow \boxed{(x, y, z) = (0, -3, 1)t \text{ for } t \in \mathbf{R}}$$

When $\lambda = -3$,

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 5 & 0 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & 5 & 0 \\ 2 & 1 & 3 \\ 3 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 - 2R_1} \begin{pmatrix} 1 & 5 & 0 \\ 0 & -9 & 3 \\ 0 & -15 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow[R_2 \to -\frac{1}{9}R_2]{R_3 \to R_3 - \frac{5}{3}R_2} \begin{pmatrix} 1 & 5 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow{R_1 \to R_1 - 5R_2} \begin{pmatrix} 1 & 0 & 5/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Longrightarrow x + \frac{5}{3}z = 0 \text{ and } y - \frac{1}{3}z = 0$$

$$\Longrightarrow \boxed{(x, y, z) = (-\tfrac{5}{3}, \tfrac{1}{3}, 1)t \text{ for } t \in \mathbf{R}}$$

When $\lambda = 4$,

$$\begin{pmatrix} -5 & 1 & 3 \\ 1 & -2 & 0 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \xrightarrow{R_1 \leftrightarrow R_2} \begin{pmatrix} 1 & -2 & 0 \\ -5 & 1 & 3 \\ 3 & 0 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow[R_3 \to R_3 - 3R_1]{R_2 \to R_2 + 5R_1} \begin{pmatrix} 1 & -2 & 0 \\ 0 & -9 & 3 \\ 0 & 6 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow[R_2 \to -\frac{1}{9}R_2]{R_3 \to R_3 + \frac{2}{3}R_2} \begin{pmatrix} 1 & -2 & 0 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\xrightarrow{R_1 \to R_1 + 2R_2} \begin{pmatrix} 1 & 0 & -2/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$$

$$\Longrightarrow x - \frac{2}{3}z = 0 \text{ and } y - \frac{1}{3}z = 0$$

$$\Longrightarrow \boxed{(x, y, z) = (\tfrac{2}{3}, \tfrac{1}{3}, 1)t \text{ for } t \in \mathbf{R}}$$

Section 9.

11. *Show that a real Hermitian matrix is symmetric. Show that a real unitary matrix is orthogonal.*

Suppose A is a real Hermitian matrix. Then $A = \bar{A}$ since A is real. ($\bar{A}$ is the matrix composed of the complex conjugates of each element of A). Also, $A = (\bar{A})^T$ since A is Hermitian. But then $A = A^T$ and thus A is symmetric.

Suppose A is a real unitary matrix. Then $A = \bar{A}$ since A is real. Also $A^{-1} = (\bar{A})^T$ since A is unitary. But then $A^{-1} = A^T$ and thus A is orthogonal. **Q.E.D.**