Differential Equations: Second Homework Assignment

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Page 87, Number 29. Problem. *During the period from 1790 to 1930, the U.S. population $P(t)$ (t in years) grew from 3.9 million to 123.2 million. Throughout this period, $P(t)$ remained close to the solution of the initial value problem*

$$\frac{dP}{dt} = 0.03135P - 0.0001489P^2, \quad P(0) = 3.9$$

(a) *What 1930 population does this logistic equation predict?*
(b) *What limiting population does it predict?*
(c) *Has this logistic equation continued since 1930 to accurately model the U.S. population?*

Let $\alpha = 0.03135$, $\beta = 0.0001489$, and $P_0 = 3.9$ so that we have $P(0) = P_0$ and

$$\begin{aligned}
\frac{dP}{dt} = \alpha P - \beta P^2 &\Longrightarrow \frac{dP}{P(\alpha - \beta P)} = dt \\
&\Longrightarrow \frac{1}{\alpha}\int\left(\frac{1}{P} + \frac{\beta}{\alpha - \beta P}\right) dP = \int dt \\
&\Longrightarrow \frac{1}{\alpha}\left[\ln|P| - \ln|\alpha - \beta P|\right] = t + A \quad \text{for any } A \in \mathbf{R} \\
&\Longrightarrow \ln\left|\frac{P}{\alpha - \beta P}\right| = \alpha t + B \quad \text{where } B = \alpha A \\
&\Longrightarrow \frac{P}{\alpha - \beta P} = e^{\alpha t + B} = e^{\alpha t}e^B = Ce^{\alpha t} \quad \text{where } C = e^B
\end{aligned}$$

Then,

$$P(0) = P_0 \Longrightarrow \frac{P_0}{\alpha - \beta P_0} = C$$

so that

$$\begin{aligned}
P = (\alpha - \beta P)Ce^{\alpha t} &\Longrightarrow P = \frac{\alpha P_0 e^{\alpha t}}{\alpha - \beta P_0} - \frac{\beta P P_0 e^{\alpha t}}{\alpha - \beta P_0} \\
&\Longrightarrow P\left(1 + \frac{\beta P_0 e^{\alpha t}}{\alpha - \beta P_0}\right) = \frac{\alpha P_0 e^{\alpha t}}{\alpha - \beta P_0} \\
&\Longrightarrow P = \frac{\alpha P_0 e^{\alpha t}}{\left(1 + \frac{\beta P_0 e^{\alpha t}}{\alpha - \beta P_0}\right)(\alpha - \beta P_0)} \\
&\Longrightarrow P = \frac{\alpha P_0 e^{\alpha t}}{\alpha - \beta P_0 + \beta P_0 e^{\alpha t}} \\
&\Longrightarrow P = \frac{\alpha P_0}{(\alpha - \beta P_0)e^{-\alpha t} + \beta P_0}
\end{aligned}$$

Therefore

$$P(t) = \frac{\alpha P_0}{(\alpha - \beta P_0)e^{-\alpha t} + \beta P_0}$$

Now,

(a) The 1930 population predicted by this logistic equation is 127 million, since:

$$\begin{aligned} P(1930 - 1790) &= P(140) \\ &= \frac{\alpha P_0}{(\alpha - \beta P_0)e^{-\alpha(140)} + \beta P_0} \\ &\sim 127 \end{aligned}$$

(b) The limiting population predicted is 210.5 million, since:

$$\begin{aligned} \lim_{t\to\infty} P(t) &= \lim_{t\to\infty} \frac{\alpha P_0}{(\alpha - \beta P_0)e^{-\alpha t} + \beta P_0} \\ &= \frac{\alpha P_0}{\beta P_0} \\ &= \frac{\alpha}{\beta} \\ &\sim 210.5 \end{aligned}$$

(c) This logistic equation has **not** continued since 1930 to accurately model the U.S. population. The actual U.S. population in 2000 was 281.422 million which significantly exceeds the limiting population predicted by the model.